REDUCTION OF BEHAVIOR OF ADDITIVE CELLULAR AUTOMATA ON GROUPS

VALERIY BULITKO

ATHABASCA UNIVERSITY
VALERIYB@ATHABASCAU.CA

ABSTRACT. A class of additive cellular automata (ACA) on a finite group is defined by an index-group $\mathfrak{g}$ and a finite field $\mathfrak{F}_p$ for a prime modulus p [1]. This paper deals mainly with ACA on infinite commutative groups and direct products of them with some non commutative p-groups. It appears that for all abelian groups, the rules and initial states with finite supports define behaviors which being restricted to some infinite regular series of time moments become significantly simplified. In particular, for free abelian groups with n generators states $V^{[t]}$ of ACA with a rule R at time moments $t = p^k, k > k_0$, can be viewed as $||R||$ copies of initial state $V^{[0]}$ moving through an n-dimensional Euclidean space. That is the behavior is similar to gliders from J.Conway's automaton *Life*. For some other special infinite series of time moments the automata states approximate self-similar structures and the approximation becomes better with time. An infinite class $\mathrm{DHC}(\mathbf{S}, \theta)$ of non-commutative p-groups is described which in particular includes quaternion and dihedral p-groups. It is shown that the simplification of behaviors takes place as well for direct products of non-commutative groups from the class $\mathrm{DHC}(\mathbf{S}, \theta)$ with commutative groups. Finally, an automaton on a non-commutative group is constructed such that its behavior at time moments $2^k, k \geq 2$, is similar to a glider gun. It is concluded that ACA on non-commutative groups demonstrate more diverse variety of behaviors comparing to ACA on commutative groups.

CONTENTS

Lt $\mathfrak{g}, \mathfrak{F}_p$ be a group (index-group) and the finite field by modulo p. An element $g \in \mathfrak{g}$ given, let g^{-1} and $\overline{g}$ denote the inverse element for g in $\mathfrak{g}$, i.e. $g\overline{g} = \overline{g}g = \mathbf{1}$ where $\mathbf{1}$ (or $\mathbf{1}_\mathfrak{g}$) is the unit of $\mathfrak{g}$.

By $v(g)$ we denote g-th component of a string v where $g \in \mathfrak{g}$. Often it is also convenient by $v(g)$ to denote $v\big|^g$. A pair $(\mathfrak{g}, \mathfrak{F}_p)$ defines the semigroup $\mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$ of strings $\{V | V(g) \in \mathfrak{F}_p, g \in \mathfrak{g}\}$ with multiplication $\boxtimes$ that is a group convolution (see references in [1])

$$(1) \qquad\qquad (V \boxtimes W)\big|^k = \sum_{s \in \mathfrak{g}} V(s) W(\overline{s}k), k \in \mathfrak{g}.$$

Here $U\big|^k$ for $U \in \mathfrak{M}, k \in \mathfrak{g}$ is another convenient notation for $U(k)$. All operations with elements of $\mathfrak{F}_p$ are performed in the field, and operations with group elements - in the group. Rules and states are elements of $\mathfrak{M}$. We use common name *vectors* for elements of $\mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$.

Let V be a current state of an additive cellular automaton (ACA) with a rule R.

Application $R * V$ of rule R to state V yields next state V' of the automaton. The next state V' of the automaton can be computed as $V' = V \boxtimes R^-$ where $(R^-)\big|^i = R(-i), i \in \mathfrak{g}$ (see [1]).[1]

Let $\operatorname{supp}(A)$ denote the set $\{g | A(g) \neq 0\}$ for a vector A. we say that vectors A, B are *disjoint* if $\operatorname{supp}(A) \cap \operatorname{supp}(B) = \emptyset$.

Let $\mathfrak{M}^f = \{V \in \mathfrak{M} : |\operatorname{supp}(V)| < \infty\}$. We call elements of $\mathfrak{M}^f$ and any automaton whose initial state and rule belong to $\mathfrak{M}^f$ *finite*.

If a group $\mathfrak{g}$ commutative, by $\mathfrak{o}$ and $+$ we denote the group unit and the group operation. As usual $\underbrace{g + \cdots + g}_{k \text{ terms}} = k \cdot g, g \in \mathfrak{g}$.

$\mathbb{I}$ is the unit of semigroup $\mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$. By $1, 0$ we denote the unit and zero of the field $\mathfrak{F}_p$. Also by $\mathbb{O}$ denote zero vector in $\mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$. $V \in \mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$ and $r \in \mathfrak{F}_p$ given, the vector rV is defined by the condition $(rV)|^g = r(V|^g), g \in \mathfrak{g}$.

1. CONVOLUTION ON GROUPS

Here are some elementary properties of $\boxtimes$ (some properties with references see in [1]). Let $\mathfrak{K}$ be the commutant of an index group $\mathfrak{g}$. We say that $A \in \mathfrak{V}$ is $\mathfrak{K}$-*correct* if $\forall g, f[g\overline{f} \in \mathfrak{K} \implies A(g) = A(f)]$.

An element $g \in \mathfrak{g}$ given, the vector $K[g]$ is defined as

$$K[g](h) = \begin{cases} 1, & \text{if } h = g \\ 0, & \text{else.} \end{cases}$$

Obviously we have $A = \sum_{g \in \operatorname{supp}(A)} A(g) K[g]$ for any $A \in \mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$.

---

[1]We distinguish the denotation "$\overline{g}$" used for reverse elements of index groups from "v^-" denoting elements of certain kind in semi-groups.

Lemma 1.   (i) $\boxtimes$ *is an associative operation.*

(ii) $K[f] \boxtimes K[g] = K[fg]$ *for any* $f, g \in \mathfrak{g}$.

(iii) $(K[g] \boxtimes A)(f) = A(\overline{g}f), \quad (A \boxtimes K[g])(f) = A(f\overline{g})$.
In particular, $K[\mathbf{1}] \boxtimes A = A \boxtimes K[\mathbf{1}] = A$.

(iv) $A \boxtimes (B + C) = A \boxtimes B + A \boxtimes C, \ \ (B + C) \boxtimes A = B \boxtimes A + C \boxtimes A$.

(v) $A^- \boxtimes B^- = (B \boxtimes A)^-$.
(vi) $A \boxtimes B = B \boxtimes A$ *for any* $A, B \in \mathfrak{V}$ *such that at least one of them is* $\mathfrak{K}$*-correct.*

(vii) *If* G *is commutative group then* $A \boxtimes B = B \boxtimes A$.

(viii) *Let* H *be a subgroup of* $\mathfrak{g}$ *and* $\mathrm{supp}(A), \mathrm{supp}(B) \subseteq H$. *Then* $\mathrm{supp}(A \boxtimes B) \subseteq H$. *In addition if* H *is commutative then* $A \boxtimes B = B \boxtimes A$.

Proof. (i) For $A, B, C \in \mathfrak{V}, g \in \mathfrak{g}$ we have

$$[(A \boxtimes B) \boxtimes C](g) = \sum_f \left(\sum_r A(r)B(\overline{r}f) \right) C(\overline{f}g) = \sum_r A(r) \sum_f B(\overline{r}f)C(\overline{f}g) =_{h:=\overline{r}f}$$

$$= \sum_r A(r) \left(\sum_h B(h)C(\overline{h}\overline{r}g) \right) = \sum_r A(r)(B \boxtimes C)(\overline{r}g) = [A \boxtimes (B \boxtimes C)](g).$$

It is possible to replace bounded variable f with h since for each fixed value of r mapping $f \mapsto \overline{r}f$ is 1-1-mapping $\mathfrak{g}$ on $\mathfrak{g}$.[2]
(ii)

$$(K[f] \boxtimes K[g])(r) = \sum_s K[f](s)K[g](\overline{s}r) = K[g](\overline{f}r) = \begin{cases} 1, & \overline{f}r = g \\ 0 & \text{otherwise.} \end{cases}$$

Since $\overline{f}r = g$ means $r = fg$ we get what we need.
(iii)

$$(K[g] \boxtimes A)(f) = \sum_s K[g](s)A(\overline{s}f) =_{\text{since } K[g]=0 \text{ if } g \neq s} A(\overline{g}f).$$

$$(A \boxtimes K[g])(f) = \sum_s A(s)K[g](\overline{s}f) =_{\text{since } K[g]=0 \text{ if } g \neq \overline{s}f} A(f\overline{g}).$$

---

[2]We use similar arguments below several times.

In particular, $(K[\imath] \boxtimes A)(g) = (A \boxtimes K[\imath])(g) = A(g)$ that is $K[\imath] \boxtimes A = A \boxtimes K[\imath] = A$

(iv) Now

$$(A \boxtimes (B + C))(g) = \sum_f A(f)(B + C)(g\overline{f}) = \sum_f A(f)B(\overline{f}g) + \sum_f A(f)C(\overline{f}g) =$$
$$(A \boxtimes B)(g) + (A \boxtimes C)(g).$$

The second identity has a similar proof.

(v) $(R^- \boxtimes Q^-)(g) = \sum_f R^-(f)Q^-(\overline{f}g) = \sum_f R(\overline{f})Q(\overline{g}f)$. If we define $w = \overline{g}f$ (for any fixed g variable w runs over $\mathfrak{g}$ while f runs over $\mathfrak{g}$) then $\overline{f} = \overline{w}\,\overline{g}$ and $\sum_f R(\overline{f})Q(\overline{g}f) = \sum_w Q(w)R(\overline{w}\,\overline{g}) = (Q \boxtimes R)(\overline{g}) = (Q \boxtimes R)^-(g)$. $\square$

(vi) By definition we can write:

$$(A \boxtimes B)(g) = \sum_f A(f)B(\overline{f}g) = \sum_r B(r)A(g\overline{r})$$
$$(B \boxtimes A)(g) = \sum_r B(r)A(\overline{r}g) = \sum_f A(f)B(g\overline{f}).$$

Now if for instance B is $\mathfrak{K}$-correct then $B(\overline{f}g) = B(g\overline{f})$ for all $g, f \in \mathfrak{g}$. This is because $g\overline{f}\,\overline{\overline{f}g}$ is a commutator. Hence $(A \boxtimes B)(g) = (B \boxtimes A)(g)$.

(vii) This is the prompt consequence of (vii) due to the fact that for commutative group $\mathfrak{K} = \{\imath\}$ in which case any vector is $\mathfrak{K}$-correct.

(viii) By definition we can write:

$$(A \boxtimes B)(g) = \sum_{f \in G} A(f)B(\overline{f}g) = \sum_{f \in H} A(f)B(\overline{f}g)$$

where if $B(\overline{f}g) \neq 0$ then $\overline{f}g \in H$. With $f \in H$ this means g must belong to H in case when $(A \boxtimes B)(g) \neq 0$.

Assume H is a commutative subgroup of $\mathfrak{g}$. In this case

$$(A \boxtimes B)(g) = \sum_{f \in \mathfrak{g}} A(f)B(\overline{f}g) = \sum_{f \in H} A(f)B(\overline{f}g) =_{\text{due to } \overline{f}g = g\overline{f}} \sum_{f \in H} A(f)B(g\overline{f}) =_{w := g\overline{f}}$$
$$\sum_{w \in H} A(\overline{w}g)B(w) = \sum_{w \in \mathfrak{g}} B(w)A(\overline{w}g) = (B \boxtimes A)(g).$$

$\square$

Corollary 1. *If $A^- = A$ then for any $n \in \mathbb{Z}^+$ we have $(A^{\boxtimes n})^- = A^{\boxtimes n}$.*

Let $\mathfrak{h}$ be a subgroup of $\mathfrak{g}$. We call vector A *right-$[a, \mathfrak{h}]$-symmetric* if

(1) $\forall g[A(g) \neq 0 \implies \exists q[g = aq]]$;
(2) $\forall g[A(ag) = A(ag^{-1})]$.

2. BAKER TRANSFORMATION FOR ADDITIVE CA ON GROUP $\mathfrak{g}$

2.1. **Case of commutative $\mathfrak{g}$.** Let us use notation $T^{\boxtimes n}$ for $\underbrace{T \boxtimes \ldots \boxtimes T}_{n \text{ terms}}$ as $T \in \mathfrak{M}$ and define $\mathbf{b}_p(T)$ by the condition

(2)
$$\mathbf{b}_p(T)|^k = \begin{cases} 0, & \text{if } \{g \in \mathfrak{g}|pg = k\} = \emptyset, \\ \sum_{i \in \{g \in \mathfrak{g}|p \cdot g = k\}} T(i), & \text{otherwise.} \end{cases}$$

Lemma 2.

$$(T_1 \boxtimes \ldots \boxtimes T_n)\Big|^k = \sum_{i_1, \ldots, i_{n-1} \in \mathfrak{g}} T_1(i_1) \cdot \ldots \cdot T_{n-1}(i_{n-1}) T_n(k - i_{1\ldots} - i_{n-1})$$

Proof. $\square$

We call a group $\mathfrak{g}$ *p-normal* if the set $\{g \in \mathfrak{g}|p \cdot g = \mathfrak{o}\}$ is finite.[3] Let $\kappa = |\mathfrak{p}|$. It is also the number of terms in sums from definition (2). Clearly for any $q \in \mathfrak{g}$

(3)
$$\exists r \in \mathfrak{g}[p \cdot r = q] \implies |\{g \in \mathfrak{g}|p \cdot g = q\}| = \kappa.$$

This is because $\{g \in \mathfrak{g}|p \cdot g = \mathfrak{o}\}$ is a subgroup of $\mathfrak{g}$.

Theorem 1. *(i)* $\mathbf{b}_p$ *is well defined on* $\mathfrak{M}^f$. *If* $\mathfrak{g}$ *is p-normal then* $\mathbf{b}_p$ *is well defined on* $\mathfrak{M}$. *(ii)* $\forall T \in \mathfrak{M}[T^{\boxtimes p} = \mathbf{b}_p(T)]$ *for any prime p and any abelian p-normal index-group $\mathfrak{g}$.*

Proof. (i) It obvious. (ii) Since

(4)
$$T^{\boxtimes p}\Big|^k = \sum_{i_1, \ldots, i_{p-1} \in \mathfrak{g}} T(i_1) \cdot \ldots \cdot T(i_{p-1}) T(k - i_1 \cdots - i_{p-1})$$

We partition the collection of $T(i_1), \ldots, T(i_{p-1}), T(k - i_1 \cdots - i_{p-1})$ on subsets consisting of elements $T(i_k)$ with equal indices i_k. Let there be r subsets whose numbers of elements are $n_1, \ldots, n_r$ where $n_1 + \cdots + n_r = p$. Then we use the fact that $multinomial(n_1, \ldots, n_r)$ is multiple of prime p for cases when $r > 1$ and $n_j > 0, j = 1, \ldots, r$. The case[4] when $i_1 = i_2 = \cdots = i_{p-1} = k - i_1 - \ldots i_{p-1}$ only remains. Thus

$$T^{\boxtimes p}\Big|^k = \sum_{i \in \{j \in \mathfrak{g}|p \cdot j = k\}} (T(i))^p$$

On the other hand $f^p = f$ for any $f \in \mathfrak{F}_p$. $\square$

We use powers of $\mathbf{b}_p$ according to the definition

(5)
$$\mathbf{b}_p^0(V) = V, \quad \mathbf{b}_p^{k+1}(V) = \mathbf{b}_p(\mathbf{b}_p^k(V)).$$

---

[3]That is the number of elements of order p is finite.

[4]Recall that operations $+, \cdot$ are defined in $\mathfrak{F}_p$ by modulo p

2.2. $\mathfrak{g}$ **is not commutative.** In general the theorem 1 is not true for non commutative groups. However it works for some special classes of rules.

Theorem 2. *Let $\mathfrak{g}$ be any p-normal[5] group, $\mathfrak{K}$ its commutant, and $\mathfrak{a}$ a commutative sugroup of $\mathfrak{g}$. If T is $\mathfrak{K}$-correct or $\operatorname{supp}(T) \subseteq \mathfrak{a}$ then $T^{\boxtimes p} = \mathbf{b}_p(T)$.*

Proof. Assume that T is $\mathfrak{K}$-correct. Now we use multiplicative notation for products of group elements and rewrite (4) as follows

$$T^{\boxtimes p}\Big|^k = \sum_{i_1,\ldots,i_{p-1}\in\mathfrak{g}} T(i_1) \cdot \cdots \cdot T(i_{p-1}) T(\overline{i_1 \ldots i_{p-1}} \, k).$$

Due to the condition that T is $\mathfrak{K}$-correct, the value of $T(\overline{i_1 \ldots i_{p-1}} \, k)$ does not depend on order of the group elements in the product $\overline{i_{p-1} \ldots i_1} \, k$: all permutations leave the product in the same adjacent class of $\mathfrak{g}$ respectively $\mathfrak{K}$. Therefore the reasoning from the proof of (ii) from theorem 1 is applicable here as well.

For the case when $\operatorname{supp}(T) \subseteq \mathfrak{a}$ for any $i \notin \mathfrak{a}$ it holds $T(i) = 0$ and thereby the sum

$$\sum_{i_1,\ldots,i_{p-1}\in\mathfrak{g}} T(i_1) \cdot \cdots \cdot T(i_{p-1}) T(\overline{i_1 \ldots i_{p-1}} \, k)$$

obviously could be restricted by elements of the commutative subgroup $\mathfrak{a}$:

$$T^{\boxtimes p}\Big|^k = \sum_{i_1,\ldots,i_{p-1}\in\mathfrak{a}} T(i_1) \cdot \cdots \cdot T(i_{p-1}) T(\overline{i_1 \ldots i_{p-1}} \, k).$$

Therefore the reasoning from theorem 1 works here as well. $\square$

For simplicity below we often use T^k instead of $T^{\boxtimes k}$.

3. DECOMPOSITION EFFECTS FOR ACA ON COMMUTATIVE GROUPS

3.1. **Case of finite commutative group.**

Decompositional effects for finite commutative groups were described in [2]. Here we reformulate the results. Let a cyclic group C_n of order n be a direct factor of an index group $\mathfrak{g}$, that is $\mathfrak{g} = G \times C_n$ for a group G. Assume that c is a generator for C_n. If non-negative integer h is the maximal number such that $p^h \mid n$, then there are p^h adjacent classes $i \cdot c + H, i = 0, \ldots, p^h - 1$, in C_n where H is the subgroup of C_n consisting of $q, q = \frac{n}{p^h}$, elements $\{jp^h \cdot c | j = 0, \ldots, q-1\}$. Any state $M \in \mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$ (we call it *global*) can be represented as a sum

$$\sum_{i=0}^{p^h-1} M_i \quad \text{of } H\text{-adjacent vectors } M_i \text{ such that} \quad M_i(g) = \begin{cases} M(g), & \text{if } g \in G \times (ic + H), \\ 0, & \text{otherwise.} \end{cases}$$

The next theorem is about decomposition of behaviors of automata for special time moments.

Theorem 3. *A rule R and an initial state M, $(R, M \in \mathfrak{M}[G \times C_n, \mathfrak{F}_p])$ given, by $\tilde{R}$ denote the vector $\mathbf{b}_p^h(R)$ and by $M_i^{[t]}$ - adjacent state for global state $M^{[t]}$ at time moment t. Then*

---

[5]This condition is not necessary if we restrict $\mathbf{b}_p$ to $\mathfrak{M}^f$.

- $\mathrm{supp}(\tilde{R}) \subseteq G \times H$;
- *for any $t \geq 1$ and $i \in [0, p^h - 1]$ state $M_i^{[t+p^h]}$ is obtained from $M_i^{[t]}$ by application of rule $\tilde{R}$;*
- *for time moments $t = jp^h, j = 1, 2, \ldots$, evolution of the global state M is assembled as*

$$M^{[t]} = \sum_{i=0}^{p^h-1} M_i^{[t]},$$

from the evolutions of adjacent states which are disjoint and could be viewed as evolving independently under action of the same (for all i) rule $\tilde{R}$.

Proof. First we have $R^{p^h} = \mathbf{b}_p^h(R) = \tilde{R}$. Then, $R * (M_i + M_j) = (M_i + M_j) \boxtimes R^- = M_i \boxtimes R^- + M_j \boxtimes R^-$ (lemma 1(iv)). Further, assume $(l, k) \in G \times C_n$. Then $(M_i \boxtimes \tilde{R}^-)|^{(l,k)} = \sum_{(g,r)} M_i(g, r)\tilde{R}^-(l - g, k - r) \neq 0$ only when $\tilde{R}(l - g, r - k) \neq 0$ for some $(g, r) \in G \times C_n$. Therefore $r - k = (r' - k')c$ where $r' = k' \pmod{p^h}$. On the other hand, by definition of M_i to have $M_i(g, r) \neq 0$ it must hold that $r \in ic + H$, i.e. $r' = i \pmod{p^h}$. From here $k \in ic + H$. This means that $\tilde{R}$ transforms vectors whose supports are subsets of some adjacent class of $G \times C_n$ respectively the subgroup $(\mathbf{1}_G, H)$ into vectors obeying the same condition. $\square$

Of course, if the factor group G of the product $G \times C_n$ has itself a direct factor C_m with m divisible by a number $p^{h'}, h' > 0$, then at least at moments of time multiple to $p^{max\{h,h'\}}$ the decomposition becomes more deep, so on.

Corollary 2. *If an index group $\mathfrak{g}$ is a finite commutative group of order p^h then for an automaton with a rule R and an initial state M it holds*

- $\mathbf{b}_p^h(R) \in \{f\mathbb{I} | f = \overline{0, p - 1}\}$;
- *if $\mathbf{b}_p^h(R) = f\mathbb{I}$ then for all $t = kp^h$, $k = 1, 2, \ldots$ it holds $M^{[t]} = f^k M$;[6]*
- *if $\mathbf{b}_p^h(R) = 0\mathbb{I}$ then for all $t \geq p^h$ we have $M^{[t]} = \mathbb{O}$.*

3.2. **Case of finitely generated free commutative groups. Gliders and iterative structures.**

We can represent any element g of a free commutative group $\mathfrak{f}_n$ with n generators $g_i, i = \overline{1, n}$, as $d_1 \cdot g_1 + \cdots + d_n \cdot g_n$ where all $d_i, i = \overline{1, n}$, are integers i.e. as a point $\chi(g)$ with integer coordinates $(d_1, \ldots, d_n)$ in n-dimensional Euclidean space $\mathbb{E}_n$. Respectively for any vector $V \in \mathfrak{M}[\mathfrak{f}_n, \mathfrak{F}_p]$ its support has the representation in $\mathbb{E}_n$ as a set $\chi(\mathrm{supp}(V)) = \{\chi(g) | g \in \mathrm{supp}(V)\}$.

Thus any $V \in \mathfrak{M}[\mathfrak{f}_n, \mathfrak{F}_p]$ can geometrically represented by a function $\mathcal{V} : \chi(\mathrm{supp}(V)) \to \mathfrak{F}_p$ according to the commutative diagram:

$$
\begin{array}{ccc}
\mathfrak{f}_n & \xrightarrow{\chi} & \mathbb{E}_n \\
V \downarrow & & \mathcal{V} \downarrow \\
\mathfrak{F}_p & = & \mathfrak{F}_p
\end{array}
$$

---

[6]Recall that operations $+, \cdot$ are defined in $\mathfrak{F}_p$ by modulo p.

In this section we do not distinguish V and its representation $\mathcal{V}$. This allows us operate with geometric images and their characteristics for rules and states.

Diameter $\delta(V)$ of a vector $V \in \mathfrak{M}[F_n, \mathfrak{F}_p]$ is the length of an edge of the minimal cube such that its edges are parallel to coordinate axes in $\mathbb{E}_n$ and it covers the set $\chi(\mathrm{supp}(V))$.

Weight $||V||$ of the vector is $|\mathrm{supp}(V)|$.[7]

Elements $V \in \mathfrak{M}[\mathfrak{f}_n, \mathfrak{F}_p]$ and $f \in \mathfrak{f}_n$ given, let $V^{\langle f \rangle}$ denote a vector s.t.

$$V^{\langle f \rangle}(g) = V(g - f), g \in \mathfrak{f}_n.$$

We see it as vector V *shifted* by f.[8] From the definition directly we have

Lemma 3. $\left[V^{\langle f \rangle)}\right]^{\langle g \rangle} = V^{\langle f+g \rangle}$.

For $V \in \mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$ *sparsity* of the vector V is the number $\sigma(V)$ such that $\sigma(V) = \frac{l}{2} + 1$ where l is the length of an edge of the maximal (respectively inclusion) cube obeying the condition that if we place its center into any $x \in \chi(\mathrm{supp}(V))$ then no other element of $\chi(\mathrm{supp}(V))$ appears in the cube.

Lemma 4. *For any rule R, any state V, and non-negative integers m, l it holds:*
(i) $\sigma(\mathbf{b}_p^m(R)) = p^m \sigma(R)$;
(ii) $\delta(\mathbf{b}_p^m(R)) = p^m(\delta(R) - 1) + 1$;
(iii) $\delta(R^l * V) \le \delta(V) + l(\delta(R) - 1)$.
In particular $\delta(\mathbf{b}_p^m(R)) \le p^m \delta(R)$ and $\delta(R^l * V) \le l\delta(R) + \delta(V)$.

Proof. This is because for a free group the baker transformation $\mathbf{b}_p$ just shifts any element $(x_1, \ldots, x_n) \in \chi(\mathrm{supp}(R))$ into a position $p(x_1, \ldots, x_n) = (px_1, \ldots, px_n)$. Therefore the distance in $\mathbb{E}_n$ between different elements v, w of $\mathrm{supp}(\mathbf{b}_p(R))$ increases by the factor p comparing to the distance between the preimages of the elements v, w. From here (i) and (ii) follow since the distances between the projections of the elements from $\chi(\mathrm{supp}(R))$ on all axes increase by the same factor.

On the basis of th edefinition of an application of a rule R to a state V and taking into account the finiteness of both R, V it is not difficult to see that $\delta(R * V) \le \delta(V) + \delta(R) - 1$. Indeed,

$$R * V|^f = \sum_g V_g R_{f-g}^- = \sum_g V_g R_{g-f}$$

and when the maximal component λ of $\chi(g - f)$ exceeds $\delta(Q)$ we get $R * V|^f = 0$. Thus we can consider only cases when $\lambda \le \delta(V) + \delta(R) - 1$. By a simple induction one can check that $\delta(R^l * V) \le \delta(V) + l(\delta(R) - 1)$. $\square$

Thus, we can say that any application of $\mathbf{b}_p$ *inflates* vectors.

We call vectors $V_i, i \in M$, *separated* from each other if the convex hulls of the sets $\{\chi(v)|v \in \mathrm{supp}(V_i)\}, i \in M$, have no common elements with each other in $\mathbb{E}_n$.

---

[7] Another and more accurate version: maximal coordinate-wise size of the set $\chi(\mathrm{supp}(V))$.

[8] Warning: $V^{\langle f \rangle}$ is not a power of V with exponent $\langle f \rangle$!

Below the following constructions with vectors from $\mathfrak{M}[\mathfrak{f}_n, \mathfrak{F}_p]$ are used. We say that a vector Y is obtained by p-*inflation* of order m from a vector X if $Y(p^m g) = X(g)$ and $\mathrm{supp}(Y) = \{p^m g | g \in \mathrm{supp}(X)\}$.

Another operation is defined as follows. A vector X and a vector Y given, assume that for any different $y, y' \in \mathrm{supp}(Y)$ vectors $X^{\langle y \rangle}, X^{\langle y' \rangle}$ are separated from each other. Then we define

$$X \uparrow Y = \sum_{y \in \mathrm{supp}(Y)} Y(y) X^{\langle y \rangle}.$$

and call the vector $X \uparrow Y$ *iteration* of the vector X by the vector Y. It is clear that the vector $X \uparrow Y$ **consists of** $||Y||$ **copies of the vector** X **separated from each other**.

From the definitions and lemma 4 it follows:

Corollary 3. *A free commutative group* $\mathfrak{f}_n$, *a modulus* p, *and a vector* $X \in \mathfrak{M}[\mathfrak{f}_n, p]$ *given, the vector* $\mathbf{b}_p^m(X)$ *is a* p-*inflation of the vector* X *of the order* m.

A basic fact used below can be formulated as follows:

Lemma 5. *Let* R, V *be a rule and an initial state of an automaton. Assume* $k = m + j + l$ *where* $l, j, m \in \mathbb{Z}^+$ *and* $\delta(V^{[l]}) \leq \sigma(R^j)$. *Then for the automaton's state* $V^{[k]}$ *at time* k *it holds*[9]

$$(6) \qquad V^{[k]} = R^m * \left(\sum_{r \in \mathrm{supp} R} R^j(r) \cdot (V^{[l]})^{\langle r \rangle} \right) = R^m * \left(V^{[l]} \uparrow R^j \right).$$

Proof. We have

$$V^{[k]} = R^k * V = R^m * (R^j * (R^l(V))) = R^m * (R^j * V^{[l]}).$$

Now, we can write

$$R^j = \sum_{r \in \mathrm{supp}(R^j)} R^j(r) K[r].$$

Therefore by lemma 1 (iii) we infer that

$$R^j * V^{[l]} = \sum_{r \in \mathrm{supp}(R^j)} R^j(r)(K[r] * V^{[l]}) = \sum_{r \in \mathrm{supp}(R^j)} R^j(r)(V^{[l]})^{\langle r \rangle}.$$

The condition $\delta(V^{[l]}) \leq \sigma(R^j)$ means that the distances between the elements of $\chi(\mathrm{supp}(R^j))$ are not lesser the size of the state $V^{[l]}$. Hence for different r the supports of $(V^{[l]})^{\langle r \rangle}$ are separated from each other. From here we arrive at

$$V^{[k]} = R^m * \left(V^{[l]} \uparrow R^j \right). \quad \square$$

Theorem 4. *A rule* R *and an initial state* V *given, let* $t = l + kp^i, k, l, i \in \mathbb{Z}^+$, *and* $p^i \sigma(R^k) \geq \delta(V^{[l]})$. *Then* $V^{[t]}$ *is an iteration of* $V^{[l]}$ *by* R^{kp^i}.

---

[9]Recall: we use R^m instead of $R^{\boxtimes m}$ and this relates to other powers of rule R.

Proof. To apply the previous lemma we set $m = 0$ and prove that $\sigma(R^{kp^i}) \geq p^i\sigma(R^k)$. For that we first represent R^{kp^i} as $\underbrace{R^k \boxtimes \ldots \boxtimes R^k}_{p^i \text{ times}} = \mathbf{b}_p^i(R^k)$. As we know from the results about $\mathbf{b}_p$ and corollary 3 $(x_1, \ldots, x_n) \in \chi(\mathrm{supp}(R^k)) \iff p^i(x_1, \ldots, x_n) \in \chi(\mathrm{supp}(\mathbf{b}_p^i(R^k)))$. Because of lemma 4(i) this means that $\sigma(R^{kp^i}) = p^i\sigma(R^k)$. $\square$

A simple sufficient condition for a state at a moment of time to be an iteration by a $\boxtimes$-power of a rule is given by the next

Corollary 4. $p^m\sigma(R^k) \geq l(\delta(R) - 1) + \delta(V) \implies V^{[l+kp^m]} = V^{[l]} \uparrow R^{kp^m}$.

Proof. Indeed on the basis of lemma 4 we can write $p^m\sigma(R^k) \geq l(\delta(R)-1)+\delta(V) \geq \delta(V^{[l]})$. Now theorem 4 allows to conclude $V^{[l+kp^m]} = V^{[l]} \uparrow R^{kp^m}$. $\square$
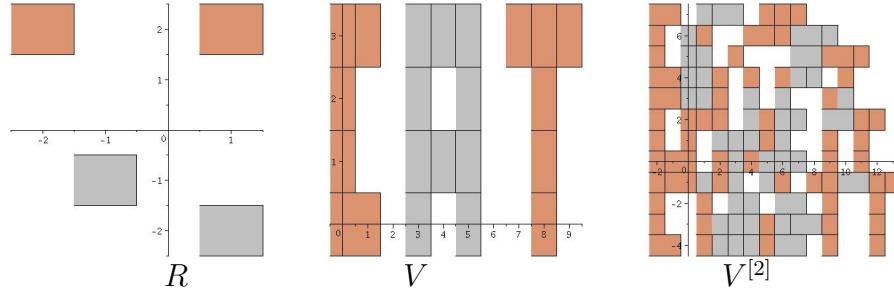

FIGURE 1. Example 1: Rule R, initial state V, and state $V^{[2]}$ at $t = 2$. Gray color corresponds to value 2 of a cell state, brown - value 1 (modulus $p = 3$).
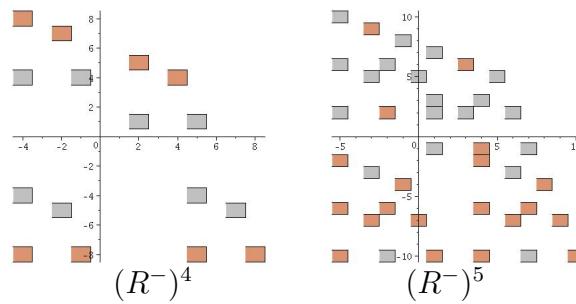

FIGURE 2. Example 1: Powers $(R^-)^4$ and $(R^-)^5$.
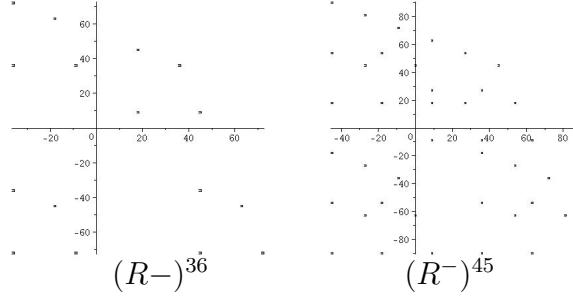
FIGURE 3. Example 1: Powers $(R^-)^{36}$ and $(R^-)^{45}$.
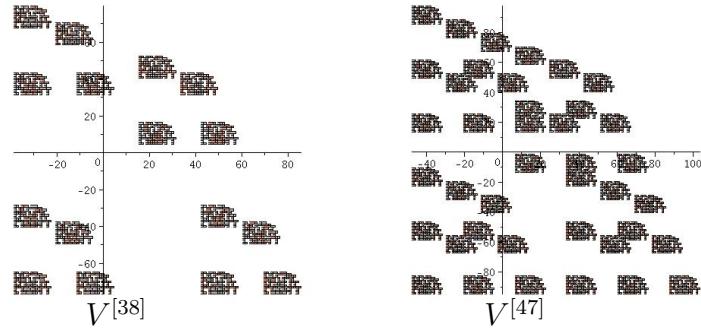
FIGURE 4. Example 1: states $V^{[38]}$ and $V^{[47]}$.

Example 1. Let $p = 3$. For the rule R and the initial state V shown on Fig. 1 we have $\delta(R) = 5, \sigma(R) = 2, \delta(V) = 10$ whereas $\delta(V^{[2]}) = 16$. Also $\sigma(R^4) = 2$ whereas $\sigma(R^5) = 1$. Let $l = i = 2, k = 4$. The powers R^4, R^5 represented in the symmetric form by $(R^-)^4, (R^-)^5$ on Fig 2.

Then $p^i \sigma(R^4) = 9 \cdot 2 = 18 > 16 = \delta(V^{[2]}) = 16$. For this case $t = l + kp^i = 38$ and theorem 4 states that $V^{[38]}$ should be the iteration of $V^{[2]}$ (see Fig. 1) by R^{36} which is a 3-inflation of R^4 of order 2. The rule $(R^-)^{36}$ (which is a symmetric to R^{36}) is shown on Fig. 3 and $V^{[38]}$ is shown on Fig. 4 (left).

The values $i = 2, k = 5, l = 2$ define the time moment $t = l + kp^i = 2 + 45 = 47 = 2 + 1200_3$ when the condition from theorem 4 is not obeyed. Indeed $p^i \sigma(R^5) = p^i = 3^2 < \delta(V[2]) = 16$. Therefore (see Fig. 4, right part) $V^{[47]}$ is not an iteration of $V^{[2]}$ by the rule R^{45} despite R^{45} is a p-inflation of R^5 of the order 2. $\square$

One more

Corollary 5. *A rule* $R \in \mathfrak{M}[\mathfrak{f}_n, \mathfrak{F}_p]$ *and a number* $m \in \mathbb{Z}^+$ *given, for any initial state* V *such that* $p^m \sigma(R) \geq \delta(V)$ *and for any integer* $k \geq m$

$$V^{[p^k]} = V \uparrow R^{p^k}$$

(i.e. the state $V^{[p^k]}$ *is an iteration of* V *by a* p-*inflation* R^{p^k} *of the rule* R *of the order* k).

Proof. Set $l = 0$ in the previous corollary. $\square$

The corollary states that if we observe at moments $t = p^k, k \geq m$, the automaton on $\mathfrak{f}_n$ defined by a rule R and an initial state V such that $p^m \sigma(R) \geq \delta(V)$ is true, then in $\mathbb{E}_n$ we will see permanent movement of $\|R\|$ copies of the initial state V away from each other as it is illustrated by the next two examples.
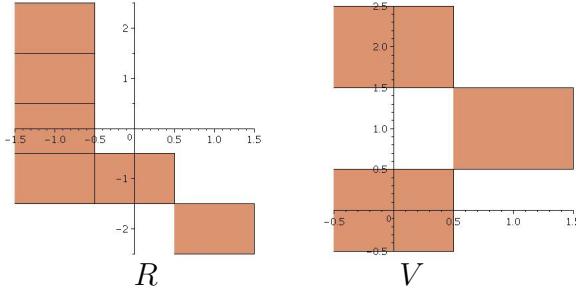
FIGURE 5. Rule R and initial state V (modulus $p = 2$).

$$t = 4 \qquad\qquad t = 8 \qquad\qquad t = 16$$
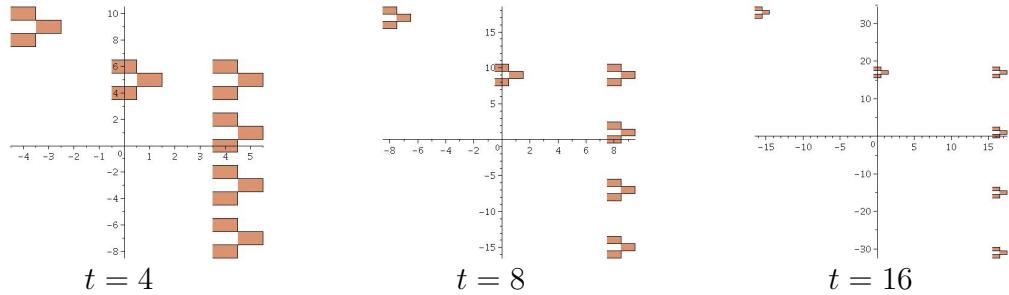
FIGURE 6. $V^{[t]}$ for rule R and initial state V shown on Fig. 5.

Example 2. Let $p = 2$. For the rule R and the intial state V shown on Fig. 5 we have $\sigma(R) = 1, \delta(V) = 2$. Therefore for each $m \geq 2$ it holds that $V^{[2^m]}$ is a 2-inflation of the order m of the intitial state by the rule. Fig. 6 shows cases $m = 2, 3, 4$. $\square$

Corollary 5 suggests an upper estimate of the number m (a threshold of inflation) starting from which we obtain inflation of the initial state for all moments of time $p^k, k \geq m$. However for some initial states V obeying the condition of the corollary it could be that an inflation happens at moments p^k for some $k < m$ as well. The next example demonstrate this.[10]
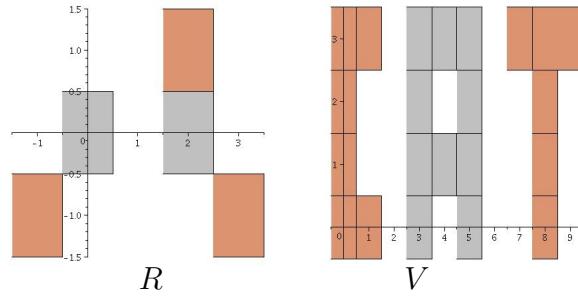
FIGURE 7. Rule R and initial state V. Gray color corresponds to the value 2 of a cell state, brown - the value 1 (the modulus $p = 3$).

$t = 9$ $\qquad$ $t = 27$ $\qquad$ $t = 81$
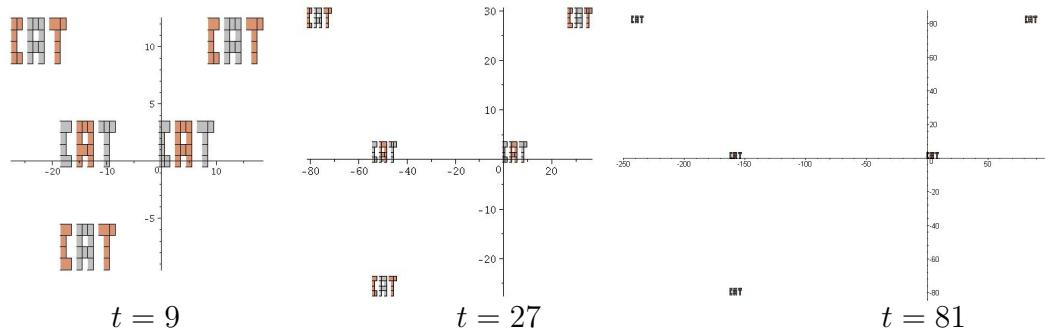
FIGURE 8. $V^{[t]}$ for the rule R and the initial state V shown on Fig. 7.

Example 3. Let $p = 3$. For the rule R and the intial state V shown on Fig. 7 we have $\sigma(R) = 1, \delta(V) = 10$. Therefore the estimate for m given by corollary 5 is 3. However for each $m \geq 2$ it holds that $V^{[3^m]}$ is a 3-inflation of the order m of the intitial state by the rule. Fig. 8 shows cases $m = 2, 3, 4$. $\square$

---

[10]More accurate estimate could be built in terms of coordinate-wise sizes for states and coordinate-wise sparsities for rules.

In logarithmic scale for space and time the behavior described by corollary 5 could be seen as a movement of $||R||$ copies of an initial state through the space away from each other. This reminds glider's flight (if we use terminology from the theory of J.Conway's automaton *Life*). Two essential differences are:

- the movement of each copy ("glider") is not an autonomous;
- the behavior is a typical does not matter what are non-zero rule and initial state.

Theorem 5. *A rule R and an initial state V given, let $l_j = \sum_{i=1}^{j} p^{m_i}, 0 < j \leq k \in \mathbb{Z}$, where $\sigma(R)p^{m_1} \geq \delta(V)$ and for each $j \in \{1, 2, \ldots, k-1\}$ it hold that $p^{m_{j+1}} > l_j \frac{\delta(R)-1}{\sigma(R)}$. Then*

$$V^{[l_k]} = (\ldots (V \uparrow \mathbf{b}_p^{m_1}(R)) \uparrow \ldots) \uparrow \mathbf{b}_p^{m_k}(R).$$

One particular choice for $m_j, j = \overline{2, k}$, could be $m_j = m_1 + (j-1)\mu$ where $p^\mu \geq \frac{\delta(R)-1}{\sigma(R)} + 1$.

Proof. When $k = 1$ this statement follows from theorem 4. Note that

$$\delta(V^{[l_j]}) \leq l_j(\delta(R) - 1) + \delta(V) \leq l_j(\delta(R) - 1) + p^{m_1}\sigma(R),$$

lemma 4(iii). Hence if

$$p^{m_{j+1}}\sigma(R) \geq l_j(\delta(R) - 1) + p^{m_1}\sigma(R)$$

then $V^{[l_{j+1}]} = V^{[l_j]} \uparrow R^{p^{m_{j+1}}}$. We can rewrite

$$p^{m_{j+1}}\sigma(R) \geq l_j(\delta(R) - 1) + p^{m_1}\sigma(R) \quad \text{as} \quad p^{m_{j+1}} \geq l_j\frac{\delta(R) - 1}{\sigma(R)} + p^{m_1}.$$

The latter inequality can be satisfied by the condition

$$\frac{p^{m_{j+1}}}{l_j} > \frac{\delta(R) - 1}{\sigma(R)} \quad \text{because} \quad \frac{p^{m_1}}{l_j} < 1 \quad \text{if} \quad j > 1.$$

Thus the "general part" of the statement is proved.

Assume $m_{j+1} = m_j + \mu, j = \in \mathbb{Z}^+$ and thereby $m_j = m_1 + (j-1)\mu$. From here

$$l_j = \sum_{i=1}^{j} p^{m_i} = p^{m_1} \sum_{i=0}^{j-1} (p^\mu)^i = p^{m_1}\frac{p^{j\mu} - 1}{p^\mu - 1}$$

Hence

$$\frac{p^{m_{j+1}}}{l_j} = \frac{p^{m_{j+1}}(p^\mu - 1)}{p^{m_1}(p^{j\mu} - 1)} = \frac{p^{m_1}p^{j\mu}(p^\mu - 1)}{p^{m_1}(p^{j\mu} - 1)} > p^\mu - 1$$

because $j > 1, \mu \geq 1, p \geq 2$. This means that we can choose μ satisfying the condition $p^\mu \geq \frac{\delta(R)-1}{\sigma(R)} + 1$. $\square$

According to this theorem the arising iterational structures approximate in a sense self-similar structures at appropriate time moments. The following examples shows the structures of level of iteration 2, 3, and 4 at the first time moments of their appearance.

$t = 20$ $t = 84$ $t = 340$
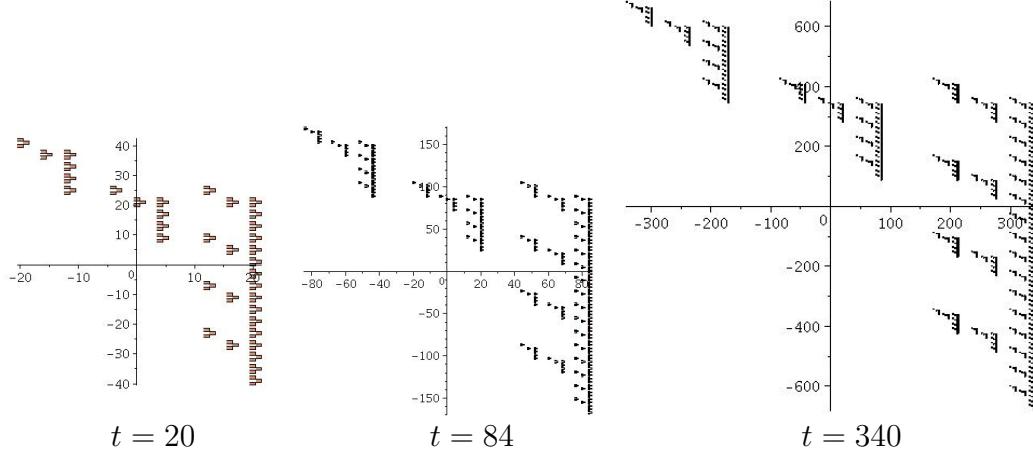
FIGURE 9. Iterational structures for $p = 2$ and R, V shown on Fig. 5.

Example 4. Fig. 9 shows the iterational structures for the data (p, R, V) from example 2 at time moments $t = 20, 84, 340$. The p-adic notations of these numbers are $10100, 1010100, 101010100$ respectively. (That is for instance $340 = 2^2 + 2^4 + 2^6 + 2^8$.) We have

$$V^{[340]} = \left[\left[\left[V \uparrow \mathbf{b}_2^2(R)\right] \uparrow \mathbf{b}_2^4(R)\right] \uparrow \mathbf{b}_2^6(R)\right] \uparrow \mathbf{b}_2^8(R)$$

and each iteration $\uparrow$ from left to right in this formula iterates the previous structure by the rule R appropriately inflated. This can lead (in a limit) to a certain kind self-similarity.

On the other hand, due to $\delta(R) = 5, \sigma(R) = 1$ theorem 5 warranties iteration structures similar to the presented on Fig. 9 only when the difference $m_{i+1} - m_i$ is equal to 3 at least. Therefore the numbers $t = 20, 84, 340$ should (according to the theorem) look like $t = 36, 292, 2340$.

This example shows that the characteristic $\delta(R)$ could be replaced with more accurate one. For example, as we noticed above, the further step could be a replacement of the defined sizes of rules and states with collections of coordinate-wise sizes. In this case more accurate consideration shows that in fact an effective size of R in this example is equal to 3. From here 2^μ must be not lesser than 3 resulting in $\mu = 2$. And the series of m_i calculated on the basis $m_1 = 2$ and $\mu = 2$ consists of exactly the same time moments $20, 84, 340, \ldots$ as on the Fig. 9. $\square$

3.3. **Cartesian products of commutative groups.**

Here the decomposition effects are combinations of the effects described above for periodical groups and free groups because we consider only finitely generated commutative groups.

One peculiarity that worthes to be noticed could be formulated as follows: the observable size of the periodical component G for product $G \times F$ with a free group F can depend on the size of the step of observations in time. In degenerated case when G is a p-group with the maximal order of its elements p^s and the length of the step is $p^k, k \geq s$, the component

G is unobservable. With more details we consider a similar case below for direct products of non-commutative 2-groups of a certain kind with commutative groups.

4. ACA ON NON-COMMUTATIVE GROUPS FOR THE CASE $p = 2$

4.1. **Representation of $\boxtimes$ by operations on sets in case $p = 2$.**

Now we consider another representation of the convolution $\boxtimes$. A modulus p and vectors $V, W \in \mathfrak{M}[\mathfrak{g}, \mathfrak{F}_p]$ given, let be

$$
(7) \qquad V \circledast W = \left\{ f : \sum_{\substack{v' \in \mathrm{supp}(V), w' \in \mathrm{supp}(W) \\ v'w'=f}} V(v')W(w') \neq 0 \,(\mathrm{mod}\, p) \right\}.
$$

Note that the result of the operation $\circledast$ is not a vector but a subset of $\mathfrak{g}$.

Lemma 6. $\mathrm{supp}(V \boxtimes W) = V \circledast W$.

Proof. By the definition of $\boxtimes$ and supp we have

$$
f \in \mathrm{supp}(V \boxtimes W) \iff \sum_{v \in \mathfrak{g}} V(v)W(\overline{v}f) \neq 0 \,(\mathrm{mod}\, p).
$$

Let $w = \overline{v}f$.

$$
f \in \mathrm{supp}(V \boxtimes W) \iff \sum_{\substack{v',w' \in \mathfrak{g} \\ f=v'w'}} V(v')W(w') \neq 0 \,(\mathrm{mod}\, p).
$$

Taking into account that $f = vw$ and $X(u) \neq 0 \iff u \in \mathrm{supp}(X)$ we can rewrite the previous equivalence as following

$$
f \in \mathrm{supp}(V \boxtimes W) \iff \sum_{\substack{v' \in \mathrm{supp}(V), w' \in \mathrm{supp}(W) \\ v'w'=f}} V(v')W(w') \neq 0 \,(\mathrm{mod}\, p). \quad \square
$$

Corollary 6. *The operation $\circledast$ is associative. In case when $A \boxtimes B = B \boxtimes A$ we have $A \circledast B = B \circledast A$.*

For the binary case ($p = 2$) vectors actually are characteristic functions of sets and therefore the operation $\circledast$ can be seen as an operation on subsets of an index group and the definition (7) could be rewritten as follows:

$$
M \circledast L = \left\{ f : |\{(v', w')|v' \in M, w' \in L, v'w' = f\}| = 1 \,(\mathrm{mod}\, 2) \right\}, \quad M, L \subseteq \mathfrak{g}.
$$

Further, in the case $p = 2$ we can consider $\circledast$ as an operation on subsets of an index group.

Corollary 7. *If $p = 2$ then $\mathrm{supp}(A \boxtimes B) = \mathrm{supp}(A) \circledast \mathrm{supp}(B)$.*

Proof. This is a direct consequence of lemma 6 and definition (7). □

One more useful tool for the case $p = 2$ is an operation $\flat : \mathcal{P}(\mathfrak{g}) \to \mathcal{P}(\mathfrak{g})$ on sets of elements of an index-group $\mathfrak{g}$ which accompanies the baker transformation. Namely for $G \subseteq \mathfrak{g}$ we define

$$\flat(G) = \{g : g \in \mathfrak{g} \ \wedge \ |\{q|q \in Q \ \wedge \ q^2 = g\} = 1| \, (\mathrm{mod} \ 2)\}.$$

As usual we accept that $\flat^0(G) = G$ and $\flat^{n+1}(G) = \flat(\flat^n(G))$.

Let $Q_i, i = \overline{1, n}$, be sets. We define

$$\biguplus_{i=1}^{n} Q_i = \{q \colon \ q \in \cup_{i=1}^{n} Q_i \ \wedge \ |\{i|q \in Q_i\}| = 1 \, (\mathrm{mod} \ 2)\}.$$

For sets Q, S we have $Q \uplus S = (Q \cup S) \setminus (Q \cap S)$.

In this section for a subset Q of a group $\mathfrak{g}$ by $\overline{Q}$ we denote the set $\{\overline{g}|g \in Q\}$ where $\overline{g}$ is the inverse element for g in $\mathfrak{g}$.

We call a subset S of an index group $\mathfrak{g}$ *abelian set* if all elements of S commute with each other in $\mathfrak{g}$.

Corollary 8. $p = 2$ *given,*
(i) for any $V \in \mathfrak{M}[\mathfrak{g}, \mathfrak{F}_2]$ such that $\mathrm{supp}(V)$ is an abelian set in $\mathfrak{g}$, it holds:

$$\sup(V \boxtimes V) = \mathrm{supp}(V) \circledast \mathrm{supp}(V) = \flat(\mathrm{supp}(V));$$

(ii) for any abelian sets Q, S

$$\flat(Q \uplus S) = \flat(Q) \uplus \flat(S),$$
$$\flat(Q \circledast S) = \flat(Q) \circledast \flat(S),$$
$$\flat(\overline{Q}) = \overline{\flat(Q)}.$$

Proof. (i) Let $S = \mathrm{supp}(V)$. We need to show that

$$S \circledast S =_{\mathrm{df}} \{ss' \colon \ |\{s''s''' \colon \ s'', s''' \in S, \ ss' = s''s'''\}| = 1 \, (\mathrm{mod} \ 2)\} =$$
$$\{s^2 \colon \ |\{(s')^2 \colon \ s' \in S, \ s^2 = (s')^2\}| = 1 \, (\mathrm{mod} \ 2)\}.$$

This follows from the fact that each pair (q, q') where $q \neq q'$, $q, q' \in S$ should be counted twice and therefore only cases to take into account are those where $s = s', s'' = s'''$.
(ii) This is because $s \in S \ \wedge \ q \in Q \implies sq = qs \ \wedge \ s^2 q^2 = (sq)^2 \ \wedge \ (\overline{q})^2 = \overline{q^2}$. □

4.2. **Groups** $\mathrm{DHC}(\mathbf{S}, \theta)$ **of order** 2^{k+1}**.**

Group $\mathrm{DHC}(\mathbf{S}, \theta)$ is built from a finite abelian group $\mathbf{S}$ and its element θ of the order not exceeding 2 by extension with an element r and defining relations: $r^2 = \theta$ and $rs = \overline{s}r$ for all $s \in \mathbf{S}$.

When $\theta = \mathbf{1}$ a group $\mathrm{DHC}(\mathbf{S}, \theta)$ is actually a generalized dihedral group $\mathrm{Dih}(\mathbf{S})$ (see [4], p. 210). And when $\mathbf{S}$ is a cyclic group of order q then $\mathrm{DHC}(\mathbf{S}, \theta)$ is a dihedral group $\mathfrak{D}_q$ with representation $\langle \{s, r\} | s^q = 1, \ r^2 = 1, \ sr = rs^{-1} \rangle$.

In case $\theta \neq 1$ a group $\mathrm{DHC}(\mathbf{S}, \theta)$ is a generalized dicyclic group $\mathrm{Dic}(\mathbf{S}, \theta)$ (see [5], p. 170, [6], p. 252).

If $\mathbf{S}$ is a cyclic group of an even order, its element θ of the order 2 is defined uniquely and we arrive at a dicyclic group $\mathrm{Dic}(\mathbf{S})$. Examples are: the quaternion group ($\mathbf{S}$ is a cyclic group of order 4) and the generalized quaternion group ($\mathbf{S}$ is cyclic group of order 8).

Some sources (see [7], p. 96) define as a generalized quaternion group $\mathfrak{Q}_n$ any generalized dicyclic group $\mathrm{Dic}(\mathbf{S})$ when $\mathbf{S}$ is a cyclic group of order 2^n, $n \geq 3$.

Due to the relations $rs = \bar{s}r$ for any $s \in \mathbf{S}$ the center $\mathfrak{Z}$ of $\mathrm{Dic}(\mathbf{S})$ includes all elements of $\mathbf{S}$ of the order not exceeding 2. In particular, $\theta \in \mathfrak{Z}$.

Theorem 6. *Let $\mathbf{S}$ be a commutative group of an order $2^k, k \in \mathbb{Z}^+$, and 2^κ be the maximal order of elements of $\mathbf{S}$. For any vector $V \in \mathfrak{M}[\mathrm{DHC}(\mathbf{S}, \theta), \mathfrak{F}_2]$ it is true that $V^{\boxtimes 2^\kappa}$ either is the unit of $M[\mathfrak{D}_{2^k}, \mathfrak{F}_2]$ or has all components equal to zero.*

Lemma 7. $(\forall V \in \mathfrak{M}[\mathrm{DHC}(\mathbf{S}, \theta), \mathfrak{F}_2])(\exists! \, C, D \subseteq \mathbf{S})[\mathrm{supp}(V) = rC \cup D]$.

Proof. From the relations $sr = r\bar{s}, s \in \mathbf{S}$, and $r^2 = \theta \in \mathbf{S}$, it follows that any element of $\mathrm{DHC}(\mathbf{S}, \theta)$ either belongs to $\mathbf{S}$ or has representation $rs, s \in \mathbf{S}$. $\square$

Lemma 8. *Let V be any fixed vector and $\mathrm{supp}(V) = rC_0 \cup D_0$ whereas $\mathrm{supp}(V^{\boxtimes 2^n}) = rC_n \cup D_n$. The following recursion holds:*

$$\begin{cases} C_n = C_{n-1} \circledast (D_{n-1} \uplus \overline{D_{n-1}}) \\ D_n = \flat(D_{n-1}) \uplus \theta(C_{n-1} \circledast \overline{C_{n-1}}). \end{cases}$$

Proof. Induction on n. For $n \geq 1$ we have

$$\mathrm{supp}(V^{\boxtimes 2^n}) = \mathrm{supp}(V^{\boxtimes 2^{n-1}} \boxtimes V^{\boxtimes 2^{n-1}}) = \mathrm{supp}(V^{\boxtimes 2^{n-1}}) \circledast \mathrm{supp}(V^{\boxtimes 2^{n-1}}) =_{[\text{lemma 6}]}$$
$$(rC_{n-1} \cup D_{n-1}) \circledast (rC_{n-1} \cup D_{n-1}) =$$
$$([rC_{n-1} \circledast D_{n-1}] \uplus [D_{n-1} \circledast rC_{n-1}]) \cup ([rC_{n-1} \circledast rC_{n-1}] \uplus [D_{n-1} \circledast D_{n-1}]) =$$
$$rC_n \cup D_n$$

where $C_n = C_{n-1} \circledast (D_{n-1} \uplus \overline{D_{n-1}})$ because $D_{n-1} \circledast rC_{n-1} = r\overline{D_{n-1}} \circledast C_{n-1} = rC_{n-1} \circledast \overline{D_{n-1}}$: $\overline{D_{n-1}}$ (as well as D_{n-1}) comutes with C_{n-1} since both are subsets of $\mathbf{S}$.

For D_n we (due to $C_{n-1}r = r\overline{C}_{n-1}$ and $r^2 = \theta$) have $D_n = \theta C_{n-1} \circledast \overline{C_{n-1}} \uplus \flat(D_{n-1})$ because in abelian subgroup $\mathbf{S}$ for any set M we have $M \circledast M = \flat(M)$ (corollary 8). $\square$

Lemma 9. *Solution to recursion from lemma 8 for C_n, D_n is:*

$$\begin{cases} C_n = C_0 \circledast \overset{n-1}{\underset{i=0}{\circledast}} (\flat^i(D_0) \uplus \overline{\flat^i(D_0)}), \\ D_n = \flat^n(D_0) \uplus \overset{n-1}{\underset{i=0}{\uplus}} \theta^{2^i} \left\{ \flat^i(C_0) \circledast \overline{\flat^i(C_0)} \circledast \overset{n-1}{\underset{j=i+1}{\circledast}} (\flat^j(D_0) \uplus \overline{\flat^j(D_0)}) \right\}, \end{cases}$$

where it is assumed that $\overset{b}{\underset{i=a}{\circledast}} M_i = \{\mathbf{1}\}$ in case $a > b$.

Proof. A parallel induction on n. For $n = 1$ the statement of the lemma is obvious taking into account the agreements about $\flat^0$ and $\overset{b}{\underset{a}{\circledast}}$ with $a > b$.

Now we note that in the expression for D_{n-1} the set

$$M_{n-1} = \biguplus_{i=0}^{n-2} \theta^{2^i} \left\{ \flat^i(C_0) \circledast \overline{\flat^i(C_0)} \circledast \underset{j=i+1}{\overset{n-2}{\circledast}} (\flat^j(D_0) \uplus \overline{\flat^j(D_0)}) \right\}$$

is symmetric, that is $g \in M_{n-1} \implies \overline{g} \in M_{n-1}$. In other words $M_{n-1} = \overline{M_{n-1}}$. From here $M_{n-1} \uplus \overline{M_{n-1}} = \emptyset$. Hence

$$D_{n-1} \uplus \overline{D_{n-1}} = \flat^{n-1}(D_0) \uplus \overline{\flat^{n-1}(D_0)}.$$

Using the recursion for C_n and the inductive hypothesis we arrive at

$$C_n = C_{n-1} \circledast (D_{n-1} \uplus \overline{D_{n-1}}) =$$
$$C_{n-1} \circledast (\flat^{n-1}(D_0) \uplus \overline{\flat^{n-1}(D_0)}) =$$
$$C_n = \left[C_0 \circledast \underset{i=0}{\overset{n-2}{\circledast}} (\flat^i(D_0) \uplus \overline{\flat^i(D_0)}) \right] \circledast (\flat^{n-1}(D_0) \uplus \overline{\flat^{n-1}(D_0)}).$$

For D_n we replace D_n in the recursion for C, D with the expression according to inductive hypothesis

$$D_n = \flat(D_{n-1}) \uplus \theta(C_{n-1} \circledast \overline{C_{n-1}}) =$$
$$\flat \left(\flat^{n-1}(D_0) \uplus \biguplus_{i=0}^{n-2} \theta^{2^i} \left\{ \flat^i(C_0) \circledast \overline{\flat^i(C_0)} \circledast \underset{j=i+1}{\overset{n-2}{\circledast}} (\flat^j(D_0) \uplus \overline{\flat^j(D_0)}) \right\} \right)$$
$$\uplus \theta(C_{n-1} \circledast \overline{C_{n-1}}) =_{\text{lemma 8(ii)}}$$
$$\flat^n(D_0) \uplus \biguplus_{i=0}^{n-2} \flat \left(\theta^{2^i} \left\{ \flat^i(C_0) \circledast \overline{\flat^i(C_0)} \circledast \underset{j=i+1}{\overset{n-2}{\circledast}} (\flat^j(D_0) \uplus \overline{\flat^j(D_0)}) \right\} \right)$$
$$\uplus \theta(C_{n-1} \circledast \overline{C_{n-1}}) =_{\text{lemma 8(ii)}}$$
$$\flat^n(D_0) \uplus \biguplus_{i=0}^{n-2} \theta^{2^{i+1}} \left\{ \flat^{i+1}(C_0) \circledast \overline{\flat^{i+1}(C_0)} \circledast \underset{j=i+1}{\overset{n-2}{\circledast}} (\flat^{j+1}(D_0) \uplus \overline{\flat^{j+1}(D_0)}) \right\}$$
$$\uplus \theta(C_{n-1} \circledast \overline{C_{n-1}}) =$$
$$\flat^n(D_0) \uplus \biguplus_{i=1}^{n-1} \theta^{2^i} \left\{ \flat^i(C_0) \circledast \overline{\flat^i(C_0)} \circledast \underset{j=i}{\overset{n-2}{\circledast}} (\flat^{j+1}(D_0) \uplus \overline{\flat^{j+1}(D_0)}) \right\} \uplus \theta(C_{n-1} \circledast \overline{C_{n-1}}) =$$
$$\flat^n(D_0) \uplus \biguplus_{i=1}^{n-1} \theta^{2^i} \left\{ \flat^i(C_0) \circledast \overline{\flat^i(C_0)} \circledast \underset{j=i+1}{\overset{n-1}{\circledast}} (\flat^j(D_0) \uplus \overline{\flat^j(D_0)}) \right\} \uplus \theta(C_{n-1} \circledast \overline{C_{n-1}}).$$

As we already know

$$\theta\left(C_{n-1} \circledast \overline{C_{n-1}}\right) =$$

$$\theta\left(\left[C_0 \circledast \overset{n-2}{\underset{i=0}{\circledast}}(\flat^i(D_0) \uplus \overline{\flat^i(D_0)})\right] \circledast \overline{\left[C_0 \circledast \overset{n-2}{\underset{i=0}{\circledast}}(\flat^i(D_0) \uplus \overline{\flat^i(D_0)})\right]}\right) =_{\text{lemma 8}}$$

$$\theta\left(C_0 \circledast \overline{C_0} \circledast \overset{n-2}{\underset{i=0}{\circledast}}\left[(\flat^i(D_0) \uplus \overline{\flat^i(D_0)}) \circledast \overline{(\flat^i(D_0) \uplus \overline{\flat^i(D_0)})}\right]\right) =$$

$$\theta\left(C_0 \circledast \overline{C_0} \circledast \overset{n-2}{\underset{i=0}{\circledast}}\flat(\flat^i(D_0) \uplus \overline{\flat^i(D_0)})\right) = \theta\left(C_0 \circledast \overline{C_0} \circledast \overset{n-2}{\underset{i=0}{\circledast}}(\flat^{i+1}(D_0) \uplus \overline{\flat^{i+1}(D_0)})\right) =$$

$$\theta\left(C_0 \circledast \overline{C_0} \circledast \overset{n-1}{\underset{i=1}{\circledast}}(\flat^i(D_0) \uplus \overline{\flat^i(D_0)})\right) = \theta\left(\flat^0(C_0) \circledast \flat^0(\overline{C_0}) \circledast \overset{n-1}{\underset{i=1}{\circledast}}(\flat^i(D_0) \uplus \overline{\flat^i(D_0)})\right).$$

It is not difficult to see that the latter expression is the term of the sum

$$\overset{n-1}{\underset{i=0}{\uplus}}\theta^{2^i}\left\{\flat^i(C_0) \circledast \overline{\flat^i(C_0)} \circledast \overset{n-1}{\underset{j=i+1}{\circledast}}(\flat^j(D_0) \uplus \overline{\flat^j(D_0)})\right\}$$

corresponding to case $i = 0$. This ends the proof of the solution of the recursion for D_n. $\square$

Let $\mathbf{S}(j)$ be a subgroup of $\mathbf{S}$ consisting of all elements from $\mathbf{S}$ whose orders does not exceed 2^j.

Lemma 10. *It holds:*
(i) $\mathbf{S}(0) = \{\mathbf{1}\}, \quad \mathbf{S}(1) \subseteq \mathfrak{Z}, \quad \mathbf{S}(\kappa) = \mathbf{S}$;
(ii) $\forall j[\,\mathbf{S}(j) \leq \mathbf{S}(j+1)\,]$;
(iii) $\forall j \in [1, \kappa]\,\forall S \subseteq \mathbf{S}(j)\,[\,\flat(S) \subseteq \mathbf{S}(j-1)\,]$.

Proof. Obvious. $\square$

Since $\flat^{\kappa-1}(D_0) \subseteq \mathfrak{Z}$ and each element $z \in \mathfrak{Z}$ satisfies $z = \overline{z}$, we get $\flat^{\kappa-1}(D_0) \uplus \overline{\flat^{\kappa-1}(D_0)} = \emptyset$ and therefore $C_\kappa = \emptyset$ as well.

By the same reason all terms in the sum

$$\overset{\kappa-1}{\underset{i=0}{\uplus}}\theta^{2^i}\left\{\flat^i(C_0) \circledast \overline{\flat^i(C_0)} \circledast \overset{\kappa-1}{\underset{j=i+1}{\circledast}}(\flat^j(D_0) \uplus \overline{\flat^j(D_0)})\right\}$$

are empty sets. This implies $D_\kappa = \flat^\kappa(D_0) \subseteq \{\mathbf{1}\}$. Thus $\text{supp}(V^{\boxtimes 2^\kappa}) \subseteq \{\mathbf{1}\}$. This means that $V^{\boxtimes 2^\kappa}$ is either zero $\mathbf{0}$ of the semigroup or its unit $\mathbf{I}$. $\square$

5. DIRECT PRODUCT OF COMMUTATIVE GROUP AND GROUPS $\text{DHC}(\mathbf{S}, \theta)$ OF ORDER 2^{k+1}

Lemma 11. *Class* $\text{DHC}(\mathbf{S}, \theta)$ *is closed under direct product.*

Proof. Let $G_i, i = 1, 2$, be two groups of the class $\text{DHC}(\mathbf{S}, \theta)$ and for both values of i the group G_i is an extension of the abelian group $\mathbf{S_i}$ with its selected element θ_i whose order does not exceed 2. The extension is done with element r_i obeying relations $r_i^2 = \theta_i$ and $r_i s = \overline{s} r_i$ for all $s \in \mathbf{S}_i$. It is easy to see that $G_1 \times G_2$ is a group of the class $\text{DHC}(\mathbf{S}, \theta)$ with the

specification $\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2, \theta = (\theta_1, \theta_2)$, and for element (r_1, r_2) it holds that $(r_1, r^2)^2 = (r_1^2, r_2^2) = (\theta_1, \theta_2)$, $(r_1, r_2)(s, s') = (r_1 s, r_2 s') = (\overline{s} r_1, \overline{s'} r_2) = \overline{(s, s')}(r_1, r_2)$. $\square$

So we can restrict ourself with direct products $G \times A$ of one finite non-commutative group G of sort $\mathrm{DHC}(\mathbf{S}, \theta)$ and a commutative group A.

For $V \in \mathfrak{M}[G \times A, \mathfrak{F}_2]$ let $\mathrm{proj}_G(V)$ denote $\{g \in G : \exists a \in A[(g, a) \in \mathrm{supp}(V)]\}$.

Theorem 7. *Let A be a commutative group and G be a 2-group of the class $\mathrm{DHC}(\mathbf{S}, \theta)$ where the maximal order of elements of $\mathbf{S}$ is 2^κ. For any rule $R \in \mathfrak{M}[G \times A, \mathfrak{F}_2]$ if $t = 0 \,(\mathrm{mod}\, 2^\kappa)$ then $\mathrm{proj}_G(R^{\boxtimes 2^\kappa}) \subseteq \{\mathbf{1}_G\}$ where $\mathbf{1}_G$ is the unit of G.*

Proof. The proof is a modification of the proof given for theorem 6 because the statement from lemma 7 holds here as well where group $\mathbf{S} \times A$ plays the same role as $\mathbf{S}$ before (we denote it by $\mathbb{S}$):

Lemma 12. $(\forall R \in \mathfrak{M}[G \times A, \mathfrak{F}_2])(\exists! \, C, D \subseteq \mathbb{S})[\mathrm{supp}(R) = rC \cup D]$.

Proof. This time we represent elements of $G \times A$ by pairs (g, a), $g \in G$, $a \in A$. Let $\mathbf{1}_S, \mathbf{1}_A, \mathbf{1}_\mathbb{S}$ be units of groups $\mathbf{S}, A, \mathbb{S}$ respectively and thereby $\mathbf{1}_\mathbb{S} = (\mathbf{1}_S, \mathbf{1}_A)$. Clearly $\mathbf{1}_\mathbb{S} = \mathbf{1}_{G \times A}$.

For elements r, θ from the definition of a group of kind $\mathrm{DHC}(\mathbf{S}, \theta)$ applied to G we build elements $\mathbf{r}, \Theta$ as $(r, \mathbf{1}_A), (\theta, \mathbf{1}_A)$ respectfully. We have of course: $s r = \mathbf{r} \overline{s}, s \in \mathbb{S}$, and $\mathbf{r}^2 = \Theta \in \mathbb{S}$. (Note that $\mathbf{r} \notin \mathbb{S}$.) From here it follows that any element of $G \times A$ either belongs to $\mathbf{S} \times A$, that is $\mathbb{S}$, or has representation $\mathbf{r} s, s \in \mathbf{S} \times A$, i.e. belongs to $\mathbf{r} \mathbb{S}$. $\square$

Thus sets C_i, D_i from the proof of theorem 6 consist of pairs (g, a) with component-wise multiplication as it is standard for direct products of groups. The operations $\circledast, \uplus, \flat$ are directly applicable for sets of pairs that are subsets of group $\mathbb{S}$. Lemma 6 and corollary 7 are applicable to subsets S, Q of commutative group $\mathbb{S}$. For corollary 8 we complete part (ii) with statement that for $Q \subseteq \mathbb{S}$ it holds $\flat(\widetilde{Q}) = \widetilde{\flat(Q)}$ where $\widetilde{L}$ denotes $\{(\overline{g}, a) | (g, a) \in L\}$, $L \subseteq \mathbb{S}$. We need $\widetilde{L}$ because $(Q \times H)\mathbf{r} = \mathbf{r}(\overline{Q} \times H)$ for $Q \subseteq \mathbf{S}, H \subseteq A$. Thus $L\mathbf{r} = \mathbf{r}\widetilde{L}$.

The following lemma generalizes lemma 8.

Lemma 13. *Let V be any fixed vector and $\mathrm{supp}(V) = \mathbf{r} C_0 \cup D_0$ whereas $\mathrm{supp}(V^{\boxtimes 2^n}) = \mathbf{r} C_n \cup D_n$. The following recursion holds:*

$$\begin{cases} C_n = C_{n-1} \circledast (D_{n-1} \uplus \widetilde{D_{n-1}}) \\ D_n = \flat(D_{n-1}) \uplus \Theta(C_{n-1} \circledast \widetilde{C_{n-1}}). \end{cases}$$

Proof. An induction on n. For $n \geq 1$ we have

$$\mathrm{supp}(V^{\boxtimes 2^n}) = \mathrm{supp}(V^{\boxtimes 2^{n-1}} \boxtimes V^{\boxtimes 2^{n-1}}) = \mathrm{supp}(V^{\boxtimes 2^{n-1}}) \circledast \mathrm{supp}(V^{\boxtimes 2^{n-1}}) =_{[\text{lemma 6}]}$$

$$(r C_{n-1} \cup D_{n-1}) \circledast (r C_{n-1} \cup D_{n-1}) =$$

$$([r C_{n-1} \circledast D_{n-1}] \uplus [D_{n-1} \circledast r C_{n-1}]) \cup ([r C_{n-1} \circledast r C_{n-1}] \uplus [D_{n-1} \circledast D_{n-1}]) =$$

$$r C_n \cup D_n$$

where $C_n = C_{n-1} \circledast (D_{n-1} \uplus \widetilde{D_{n-1}})$ because $D_{n-1} \circledast \mathbf{r} C_{n-1} = r \widetilde{D_{n-1}} \circledast C_{n-1} = \mathbf{r}(\widetilde{D_{n-1}} \circledast C_{n-1}) = \mathbf{r}(C_{n-1} \circledast \widetilde{D_{n-1}}) = \mathbf{r} C_{n-1} \circledast \widetilde{D_{n-1}}$. We used the fact that $\widetilde{D_{n-1}}$ (as well as D_{n-1}) comutes with C_{n-1} since both are subsets of $\mathbb{S}$.

For D_n we (due to $C_{n-1}\mathbf{r} = \mathbf{r}\widetilde{C_{n-1}}$ and $\mathbf{r}^2 = \Theta$) have $D_n = \Theta C_{n-1} \circledast \widetilde{C_{n-1}} \uplus \flat(D_{n-1})$ because in abelian subgroup $\mathbb{S}$ for any set M we have $M \circledast M = \flat(M)$ (corollary 8). $\square$

Further, lemma 9 holds in form

Lemma 14. *Solution to recursion from lemma 13 for C_n, D_n is:*

$$
\begin{cases}
C_n = C_0 \circledast \overset{n-1}{\underset{i=0}{\circledast}} (\flat^i(D_0) \uplus \widetilde{\flat^i(D_0)}), \\[2mm]
D_n = \flat^n(D_0) \uplus \overset{n-1}{\underset{i=0}{\uplus}} \Theta^{2^i} \left\{ \flat^i(C_0) \circledast \widetilde{\flat^i(C_0)} \circledast \overset{n-1}{\underset{j=i+1}{\circledast}} (\flat^j(D_0) \uplus \widetilde{\flat^j(D_0)}) \right\},
\end{cases}
$$

where it is assumed that $\overset{b}{\underset{i=a}{\circledast}} M_i = \{\mathbf{1}\}$ in case $a > b$.

Proof. The transformations from the proof of lemma 9 could be repeated for $D_0, C_0 \subseteq \mathbb{S}$ with replacement of $\overline{}, r, \theta$ by $\widetilde{}, \mathbf{r}, \Theta$ respectively. $\square$

The final part of the proof is based on lemma 10 and the following fact to formulate which we denote $\{s | \exists a \in A[(s,a) \in Q]\}$ where $Q \subseteq \mathbb{S}$ by $\mathrm{proj}_S Q$.

Lemma 15. *If $\mathrm{proj}_S(\flat^m(D_0)) \subseteq \mathfrak{Z}$ then $\flat^m(D_0) \uplus \widetilde{\flat^m(D_0)} = \emptyset$.*

Proof. First we notice that $s \in \mathfrak{Z}(\mathbf{S})) \implies s^2 = \mathbf{1}_S$. Indeed, on one hand we have $rs = sr$ when $s \in \mathfrak{Z}$, on the other hand $\bar{s}r = rs$.

Since $\flat^m(D_0)$ and $\widetilde{\flat^m(D_0)}$ are sets and for each $(s,a) \in \flat^m(D_0)$, there exists $(\bar{s}, a) \in \widetilde{\flat^m(D_0)}$. And yet $\bar{s} = s$. $\square$

Thus $C_\kappa = \emptyset$ and all terms in sum

$$
\overset{\kappa-1}{\underset{i=0}{\uplus}} \Theta^{2^i} \left\{ \flat^i(C_0) \circledast \widetilde{\flat^i(C_0)} \circledast \overset{\kappa-1}{\underset{j=i+1}{\circledast}} (\flat^j(D_0) \uplus \widetilde{\flat^j(D_0)}) \right\}
$$

are empty sets. This implies $\mathrm{proj}_G(D_\kappa) = \mathrm{proj}_S(D_\kappa) = \flat^\kappa(\mathrm{proj}_S(D_0)) \subseteq \{\mathbf{1}_S\}$. Thus $\mathrm{proj}_G(\mathrm{supp}(R^{\boxtimes 2^\kappa})) \subseteq \{\mathbf{1}_G\}$, i.e. $R^{\boxtimes 2^\kappa}$ is either zero $\mathbf{0}$ or $\mathrm{proj}_G(\mathrm{supp}(R^{\boxtimes 2^\kappa})) = \{\mathbf{1}_G\}$. This means that the rule $R^{\boxtimes 2^\kappa}$ acts only on subgroup A. $\square$

Corollary 9. *Let A be a commutative group and G - a 2-group of the class $\mathrm{DHC}(\mathbf{S}, \theta)$ where the maximal order of elements of $\mathbf{S}$ is 2^κ. For any intial state V and rule R if $t = 0 \pmod{2^\kappa}$ then $\mathrm{proj}_G(V^{[t]})$ either is equal to $\mathrm{proj}_G(V)$ or is empty.*

Example 5. In this example A is a free abelian group with two generator and G - the group of quaternions of order 8, so, modulus p is equal to 2. The number κ for G is equal to 2 because 4 is the maximal order of elements in quaternions group.

Fig. 10 represents initial state V with $\mathrm{proj}_G(V) = \{\mathbf{1}_G\}$ (so, it is two-dimentional in a clear sense) and rule R with $\mathrm{supp}(R) = \{(-1,-1,2), (1,-2,2), (-2,2,1), (1,2,3)\}$. Projections of $\mathrm{supp}(R)$ on both factors A, G of the product $G \times A$ are not trivial.

Accordingly to corollary 9 on steps whose numbers are multiple $4(=p^\kappa)$ states must be empty or project into $\{\mathbf{1}_G\}$.
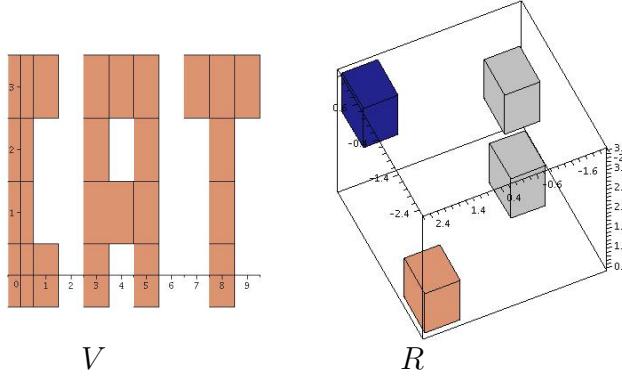
FIGURE 10. Example 5: Initial state V and rule R (modulus $p = 2$).
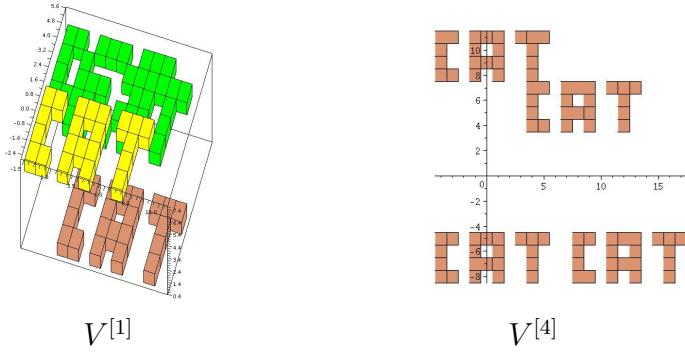
FIGURE 11. Example 5: States on step 1 (left) and 4 (right).

Fig. 11 represents states on steps 1 and 4 whereas states on steps 11 and 12 are shown on Fig. 12. We can see that states on steps 4 and 12 are "flat". □

Whether states project into $\{1_G\}$ at moments multiple to 2^κ or become empty starting from $t = 2^\kappa$, it depend on the given rule.

Example 6. Here $p = 2$, G is quaternion group of order 8, and the initial state V is the same as in example 5, whereas rule R is shown on the left part of Fig. 13. $\mathrm{supp}(R) = \{(1, 2, 1), (1, 1, 2), (1, 1, 3), 91, 2, 3), (-1, -2, 5)\}$. The right part of Fig. 13 shows $V^{[3]}$. The state $V^{[4]}$ and all next states appear to be empty. □

It appears that ACA on groups $G \times A$ (where A is abelian) demonstrate similar behavior also for some non-commutative p-groups G not included into the class $\mathrm{DHC}(\mathbf{S}, \theta)$.

Example 7. Let $p = 3$. Fig. 14, Fig. 15 illustrate the case when G is a noncommutative groups of order 27 with the identity $x^3 = 1$ (Burnside p^3-group, see [8], p.145, case II(v). A table for
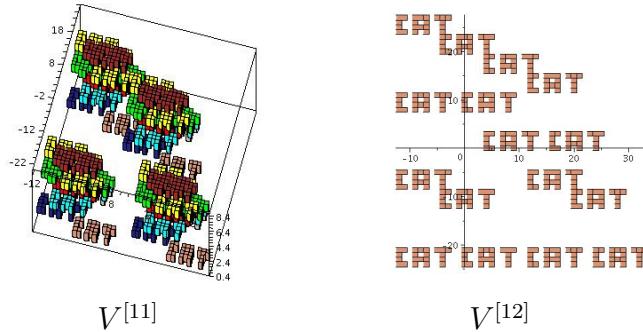
FIGURE 12. Example 5: States on step 11 (left) and 12 (right).
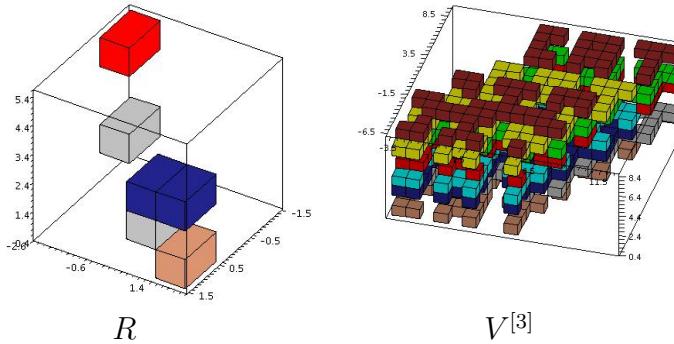
FIGURE 13. Example 6: Rule (left) and state on step 3 (right).

the group see Appendix). The description of rule R as a vector is the following:

$$R(-1,-1,2) = R(-2,2,1) = R(0,2,1) = 1$$
$$R(1,2,3) = R(-1,-2,2) = 2.$$

$\square$

Thus the results above could be extended to some other modules p and direct products of abelian groups with some other non-commutative p-groups.

6. SOME OTHER NON-COMMUTATIVE GROUPS: GLIDER GUNS

ACA on non-commutative index groups could demonstrate behaviors very different from the behaviors of ACA on abelian groups. Some simple and bright examples remind glider guns from the scope of behavious of the automaton *Life* of J.Conway.

Here we consider an example for a ACA on index group $\mathrm{DIH}3 \times \mathbb{Z}$ over field $\mathfrak{F}_2$. It is convenient to visualize automata states. For that we represent the group $\mathrm{DIH}3$ (see [3]) by
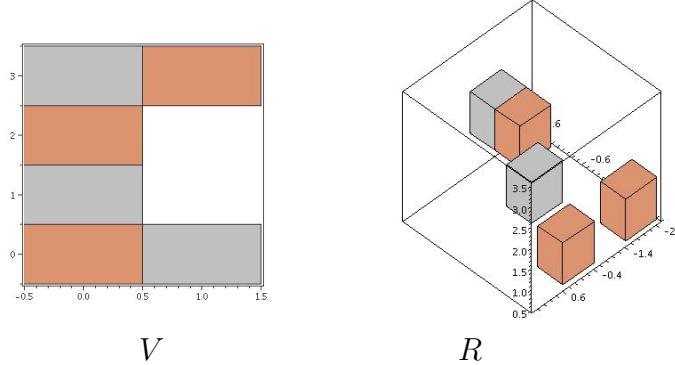
FIGURE 14. Group G (see Remark) of order 27, modulus $p = 3$: Initial state V and rule R. Brown color corresponds value 1 of a cell, gray - value 2.
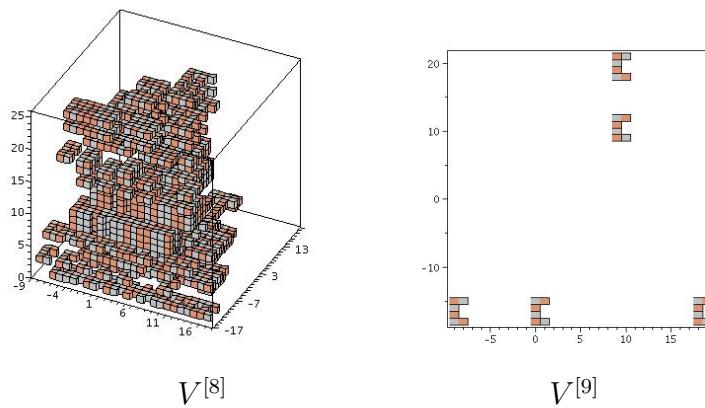
FIGURE 15. Group G (see Remark) of order 27, modulus $p = 3$: states on step 8 (left) and 9 (right).

elements $1, 2, 3, 4, 5, 6$ and multiplication table

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 4 & 5 & 1 & 6 & 3 \\
3 & 6 & 1 & 5 & 4 & 2 \\
4 & 1 & 6 & 2 & 3 & 5 \\
5 & 3 & 2 & 6 & 1 & 4 \\
6 & 5 & 4 & 3 & 2 & 1
\end{bmatrix}
$$

The whole group $\mathrm{DIH3} \times \mathbb{Z}$ can be represented as a band $\{(x, y) \colon x \in \{1, 2, 3, 4, 5, 6\}, y \in \mathbb{Z}\}$ on Eucledian plane with component-wise group operation on pairs.
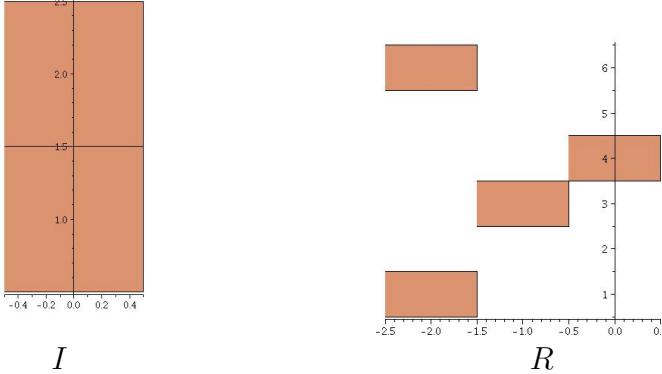
FIGURE 16. Group $\mathrm{DIH}3 \times \mathbb{Z}$, modulus $p = 2$: Initial state I and rule R.
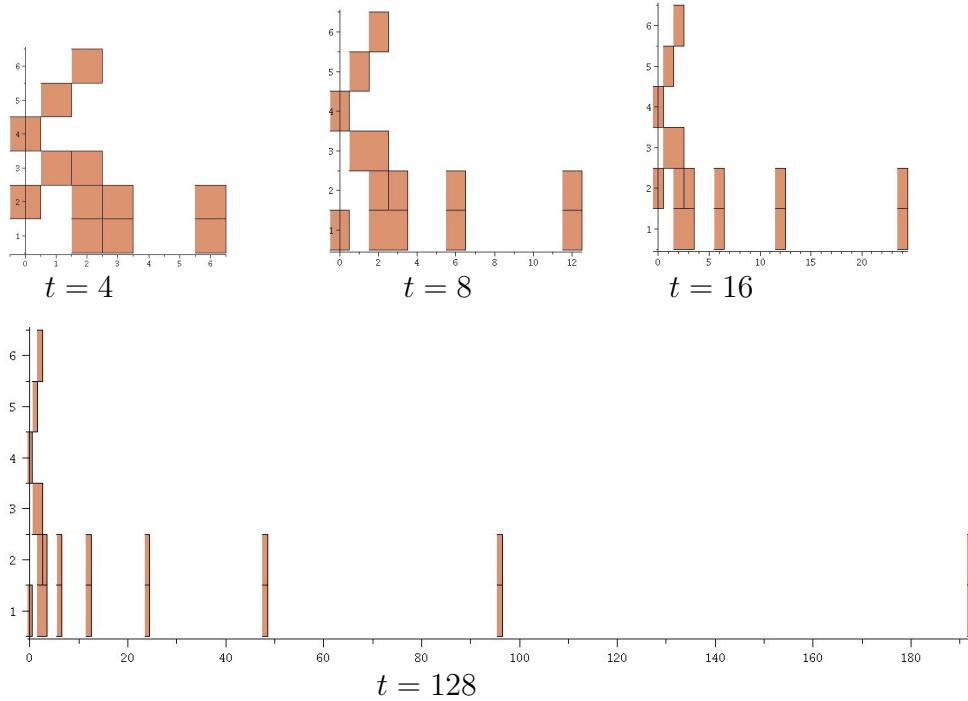
FIGURE 17. States $I_0^{[t]}$ at time moments 4,8,16,128.

Our ACA is defined by rule R and initial state I (shown on Fig. 16). States $I^{[t]}$ at time moments $t = 8, 16, 128$ are shown by Fig. 17. We see that $I^{[t]}$ for these time moments are built of subvectors (configurations) of three sorts I, B, C. (Configurations B, C are shown on Fig. 18.) The configurations can occupy different positions, and we record a position of a configuration $X \in \{I, B, C\}$ by integer $\min(\mathrm{proj}_{\mathbb{Z}}(X))$. That is X_j denote a configuration
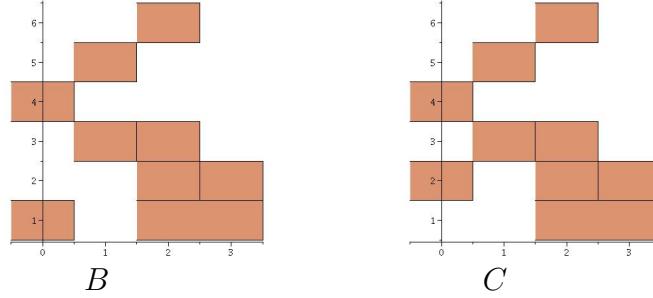
FIGURE 18. States B and C.

of kind X occupying a position in the band such that the least number of its projection on the group factor $\mathbb{Z}$ is equal to $j \in \mathbb{Z}$. In particular $I = I_0$.

In these terms Fig. 16 (left side) shows I_0 and Fig. 18 shows B_0, C_0. We have

$$\text{(8)} \qquad \text{supp}(I_0) = \{(0,1),(0,2)\},$$

$$\text{(9)} \quad \text{supp}(B_0) = \{(0,1),(0,4),(1,3),(1,5),(2,1),(2,2),(2,3),(2,6),(3,1),(3,2)\},$$

$$\text{(10)} \quad \text{supp}(C_0) = \{(0,2),(0,4),(1,3),(1,5),(2,1),(2,2),(2,3),(2,6),(3,1),(3,2)\}.$$

Due to $p = 2$ for any $z \in \mathbb{Z}$ it holds:

$$\text{(11)} \qquad\qquad I_z + B_z + C_z = 0.$$
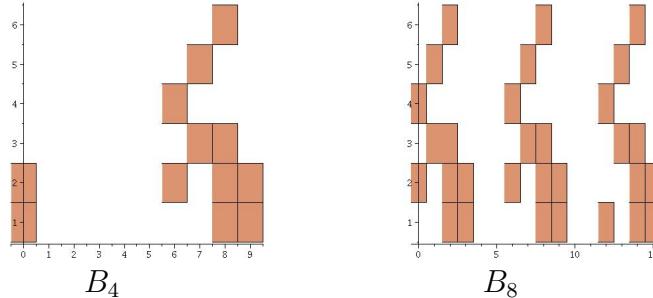
FIGURE 19. States B_4 and B_8.

Taking into account that as vectors all $B_i, C_j, I_k, i, j, k \in \mathbb{Z}$, are defined on complete group $\text{DIH3} \times \mathbb{Z}$, we can describe some states of automata in terms of sum by modulo $p, p = 2$, vectors B_i, C_j, I_k. For example, $I^{[8]}, I^{[16]}, I^{[128]}$ have the following representations:

$$I^{[8]} = B_0 + I_6 + I_{12},$$
$$I^{[16]} = C_0 + I_6 + I_{12} + I_{24},$$
$$I^{[128]} = B_0 + I_6 + I_{12} + I_{24} + I_{48} + I_{96} + I_{192}.$$
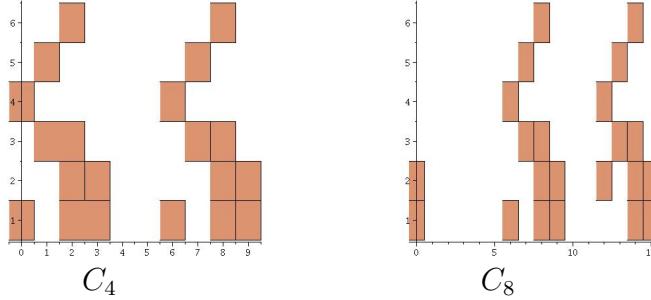
FIGURE 20. States C_4 and C_8.

Theorem 8. *For additive ACA on the group* $\mathrm{DIH3} \times \mathbb{Z}$ *with initial state* I_0 *and rule* R *it holds*

$$(12) \qquad I_0^{[2^k]} = \begin{cases} B_0 + \sum_{i=2}^{k} I_{6 \cdot 2^{i-2}}, & \textit{if } k \textit{ is odd}, \\ C_0 + \sum_{i=2}^{k} I_{6 \cdot 2^{i-2}}, & \textit{if } k \textit{ even}, \end{cases}$$

for all $k \in \mathbb{Z}, k \geq 2$.

Proof. For configuration X let $s^k(V)$ denote a shift of it on k positions right along the axis $\mathbb{Z}$. That is $z \in \mathrm{supp}(\mathrm{proj}_{\mathbb{Z}}(s^k(X))) \iff z + k \in \mathrm{supp}(\mathrm{proj}_{\mathbb{Z}}(X))$. Then by the definition of ACA we have $R * (s(X)) = s(R * X)$, in other words operations $R * \cdot$ and $s(\cdot)$ commute.

Now we formulate two more statements ($k \geq 2$):

$$(13) \qquad B_0^{[2^k]} = \begin{cases} C_0 + C_6 + \sum_{i=3}^{k} B_{6 \cdot 2^{i-2}}, & \text{if } k \text{ is odd}, \\ I_0 + C_6 + \sum_{i=3}^{k} I_{6 \cdot 2^{i-2}}, & \text{if } k \text{ even}, \end{cases}$$

and

$$(14) \qquad C_0^{[2^k]} = \begin{cases} I_0 + B_6 + \sum_{i=3}^{k} C_{6 \cdot 2^{i-2}}, & \text{if } k \text{ is odd}, \\ B_0 + B_6 + \sum_{i=3}^{k} I_{6 \cdot 2^{i-2}}, & \text{if } k \text{ even}, \end{cases}$$

and prove (12), (13), (14) by a parallel induction on k.

Basis: $k = 2, 3$. Correctness of the statements (13) and (14) has been demonstrated by direct calculations whose results are presented on Figures 19, 20.

Inductive step: $k \to k + 1$. We start with $I_0^{[2^{k+1}]}$ and odd k. Note, that by the definition $X^{[i]}$ is the state at time i of a ACA with an initial state X. Therefore $X^{[k+j]} = \left(X^{[k]} \right)^{[j]}$ for any

$k, j \in \mathbb{Z}^+$. Thus

$$I_0^{[2^{k+1}]} = \left(I_0^{[2^k]}\right)^{[2^k]} = \left(B_0 + \sum_{i=2}^{k} I_{6\cdot 2^{i-2}}\right)^{[2^k]} = B_0^{[2^k]} + \sum_{i=2}^{k} I_{6\cdot 2^{i-2}}^{[2^k]} =$$

(at this point we use hypothesis for $B_0^{[2^k]}$ and $I_{6\cdot 2^{i-2}}^{[2^k]}$)

$$= C_0 + C_6 + \sum_{i=3}^{k} B_{6\cdot 2^{i-2}} + \sum_{i=2}^{k} \left(B_{6\cdot 2^{i-2}} + \sum_{j=2}^{k} I_{6\cdot 2^{i-2}+6\cdot 2^{j-2}}\right) =$$

(we cancell equal terms by modulo 2. In particular, terms $I_{6\cdot 2^{i-2}+6\cdot 2^{j-2}}$ with $i \neq j$ disappear)

$$= C_0 + C_6 + B_6 + \sum_{i=2}^{k} I_{6\cdot 2^{i-1}} = C_0 + I_6 + \sum_{i=3}^{k+1} I_{6\cdot 2^{i-2}} =$$

$$C_0 + \sum_{i=2}^{k+1} I_{6\cdot 2^{i-2}}.$$

Similarly for even k we obtain

$$I_0^{[2^{k+1}]} = \left(C_0 + \sum_{i=2}^{k} I_{6\cdot 2^{i-2}}\right)^{[2^k]} = B_0^{[2^k]} + \sum_{i=2}^{k} I_{6\cdot 2^{i-2}}^{[2^k]} =$$

(at this point we use the hypothesis for $C_0^{[2^k]}$ and $I_{6\cdot 2^{i-2}}^{[2^k]}$)

$$= B_0 + B_6 + \sum_{i=3}^{k} C_{6\cdot 2^{i-2}} + \sum_{i=2}^{k} \left(C_{6\cdot 2^{i-2}} + \sum_{j=2}^{k} I_{6\cdot 2^{i-2}+6\cdot 2^{j-2}}\right) =$$

$$= B_0 + B_6 + C_6 + \sum_{i=2}^{k} I_{6\cdot 2^{i-1}} = B_0 + I_6 + \sum_{i=3}^{k+1} I_{6\cdot 2^{i-2}} =$$

$$B_0 + \sum_{i=2}^{k+1} I_{6\cdot 2^{i-2}}.$$

Now we consider $B_0^{[2^{k+1}]}$ for odd k:

$$B_0^{[2^{k+1}]} = \left(C_0 + C_6 + \sum_{i=3}^{k} B_{6\cdot 2^{i-2}} \right)^{[2^k]} = C_0^{[2^k]} + C_6^{[2^k]} + \sum_{i=3}^{k} B_{6\cdot 2^{i-2}}^{[2^k]} =$$

$$\left\{ I_0 + B_6 + \sum_{i=3}^{k} C_{6\cdot 2^{i-2}} \right\} + \left\{ I_6 + B_{12} + \sum_{i=3}^{k} C_{6+6\cdot 2^{i-2}} \right\} + \sum_{i=3}^{k} \left(C_{6\cdot 2^{i-2}} + C_{6+6\cdot 2^{i-2}} + \right.$$

$$\left. \sum_{j=3}^{k} B_{6\cdot 2^{i-2}+6\cdot 2^{j-2}} \right) = I_0 + I_6 + B_6 + B_{12} + \sum_{i=3}^{k} B_{6\cdot 2^{i-1}} =$$

$$I_0 + C_6 + B_{12} + \sum_{i=4}^{k+1} kB_{6\cdot 2^{i-2}} = I_0 + C_6 + \sum_{i=3}^{k+1} kB_{6\cdot 2^{i-2}}$$

And for even k:

$$B_0^{[2^{k+1}]} = \left(I_0 + C_6 + \sum_{i=3}^{k} B_{6\cdot 2^{i-2}} \right)^{[2^k]} = I_0^{[2^k]} + C_6^{[2^k]} + \sum_{i=3}^{k} B_{6\cdot 2^{i-2}}^{[2^k]} =$$

$$\left\{ C_0 + \sum_{i=3}^{k} I_{6\cdot 2^{i-2}} \right\} + \left\{ B_6 + B_{12} + \sum_{i=3}^{k} C_{6+6\cdot 2^{i-2}} \right\} + \sum_{i=3}^{k} \left(I_{6\cdot 2^{i-2}} + C_{6+6\cdot 2^{i-2}} + \right.$$

$$\left. \sum_{j=3}^{k} B_{6\cdot 2^{i-2}+6\cdot 2^{j-2}} \right) = C_0 + I_6 + B_6 + B_{12} + \sum_{i=3}^{k} B_{6\cdot 2^{i-1}} =$$

$$C_0 + C_6 + B_{12} + \sum_{i=4}^{k+1} kB_{6\cdot 2^{i-2}} = C_0 + C_6 + \sum_{i=3}^{k+1} kB_{6\cdot 2^{i-2}}.$$

It is possible to get the expression for $C^{[2^{k+1}}$ directly or by the symmetry

$$(k \text{ odd}, B, C) \iff (k \text{ even}, C, B)$$

which holds for equations (13), (14). That is if we replace words "odd", with "even" and vice versa, and all occurences of B (C) replace with C (B) respectively, the system does not change. With this symmetry the induction step for C can be obtained from the inductive step for B. $\square$

Despite the nature of the described behavior differs from the case of the automaton *Life*, additive glider guns for non-commutative groups can be viewed as analogues of glider guns from *Life*. Also as it follows from the results for commutative groups, no additive automata exist on commutative index-groups that are able to demonstrate the behaviour described in theorem 8. On the other hand, any commutative group can be extended in a way to a non-commutative group. Therefore obviously for any additive CA A on abelian group there exists an appropriate non-commutative group and suitable additive CA on it that represents the behavior of A.

Corollary 10. *Class behavior patterns of additive CA on non-commutative groups is essentially wider the class of behavior patterns of additive CA on commutative groups.*

7. CONCLUSION

Thus for ACA on groups, behavior can simplify essentially if observed in time moments from a special infinite series. Moreover, the size of observed parts of finite direct factors and even the observed topology (opportunity to notice some dimentions) can depend on the length of time step of observations.

As we noted above some typical behaviors of ACA on commutative groups in special time moments could be viewed as gliders' flights from the well known automaton *Life* of J. Conway. Another pattern of behavior for the latter automaton is gilder gun. These two phenomena play important role in the theory of the automaton *Life*. Additive cellular automata on some non-commutative groups can demonstrate behaviors simulating gliders and glider guns.

REFERENCES

[1] V.K. Bulitko, "Isomorphism of Additive CA on finite groups", arXiv:0812.0111v1 [nlin.CG], 29 Nov 2008.

[2] V.K.Bulitko, B.Voorhees, V.V. Bulitko, "Discrete Baker Transformations for Linear Cellular Automata Analysis", Journal of Cellular Automata, **1**, 2006, 40-70.

[3] H.S.M. Coxeter, W.O.J. Moser, "Generators and relations for discrete groups", 3rd ed., Springer-Verlag, 1972.

[4] J.S. Rose, "A course on group theory". Cambridge university press, 1978.

[5] G.A.Miller, H.F. Blichenfeldt, L.E. Dickson, "Theory and Applications of Finite Groups", N.-Y., G.E.Stetchert & Co., 1938.

[6] W.R. Scott, "Group Theory", Dover edition, 1987.

[7] J.J. Rotman, "The Theory of Groups. An Introduction", Allyn and Bacon, Inc., Boston, second printing, 1966.

[8] W. Burnside, "Theory of Groups of Finite Order", Dover Publication, second edition, 1955.

APPENDIX

This is the table for the group G from example 7.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
2	4	5	1	9	8	11	16	3	14	17	18	15	20	24	6	7	22	25	10	23	12	26	13	27	21	19
3	6	7	10	13	12	1	17	18	15	21	2	19	22	4	24	25	23	5	11	20	27	9	26	8	16	14
4	1	9	2	3	16	17	6	5	20	7	22	24	10	13	8	11	12	27	14	26	18	21	15	19	23	25
5	8	11	14	15	18	2	7	22	24	23	4	25	12	1	13	27	26	9	17	10	19	3	21	16	6	20
6	10	13	3	18	17	21	24	7	22	25	23	4	11	26	12	1	27	8	15	9	2	16	19	14	20	5
7	12	1	15	19	2	3	25	23	4	20	6	5	27	10	26	8	9	13	21	11	14	18	16	17	24	22
8	14	15	5	22	7	23	13	11	12	27	26	1	17	21	18	2	19	16	24	3	4	6	25	20	10	9
9	16	17	20	24	22	4	11	12	13	26	1	27	18	2	15	19	21	3	7	14	25	5	23	6	8	10
10	3	18	6	7	24	25	12	13	11	1	27	26	15	19	17	21	2	14	22	16	23	20	4	5	9	8
11	18	2	24	25	4	5	27	26	1	10	8	9	19	14	21	16	3	15	23	17	20	22	6	7	13	12
12	15	19	7	23	25	20	26	1	27	8	9	10	21	16	2	3	14	17	4	18	6	24	5	22	11	13
13	17	21	22	4	23	6	1	27	26	9	10	8	2	3	19	14	16	18	25	15	5	7	20	24	12	11
14	5	22	8	11	13	27	18	15	17	2	19	21	24	25	7	23	4	20	12	6	26	10	1	9	3	16
15	7	23	12	1	26	8	2	19	21	3	14	16	4	5	25	20	6	22	27	24	9	11	10	13	18	17
16	20	24	9	12	11	26	15	17	18	19	21	2	7	23	22	4	25	6	13	5	1	8	27	10	14	3
17	22	4	13	27	1	9	19	21	2	14	16	3	25	20	23	6	5	24	26	7	10	12	8	11	15	18
18	24	25	11	26	27	10	21	2	19	16	3	14	23	6	4	5	20	7	1	22	8	13	9	12	17	15
19	25	20	27	10	9	12	3	14	16	18	15	17	6	7	5	22	24	23	8	4	13	1	11	26	2	21
20	9	12	16	17	15	19	22	24	7	4	25	23	13	27	11	26	1	10	18	8	21	14	2	3	5	6
21	23	6	26	8	10	13	14	16	3	15	17	18	5	22	20	24	7	4	9	25	11	27	12	1	19	2
22	13	27	17	21	19	14	23	4	25	6	5	20	26	8	1	9	10	11	2	12	16	15	3	18	7	24
23	26	8	21	16	14	15	20	6	5	24	7	22	9	12	10	13	11	1	3	27	17	19	18	2	25	4
24	11	26	18	2	21	16	4	25	23	5	20	6	1	9	27	10	8	12	19	13	3	17	14	15	22	7
25	27	10	19	14	3	18	5	20	6	22	24	7	8	11	9	12	13	26	16	1	15	2	17	21	4	23
26	21	16	23	6	20	24	10	8	9	13	11	12	3	18	14	15	17	2	5	19	7	25	22	4	27	1
27	19	14	25	20	5	22	9	10	8	12	13	11	16	17	3	18	15	21	6	2	24	4	7	23	1	26